Channel Assignment with Separation in Wireless Networks Based on Regular Plane Tessellations

Alan A. BERTOSSI [a] and Cristina M. PINOTTI [b,1]

[a] *Department of Computer Science, University of Bologna, Italy*

[b] *Department of Computer Science and Mathematics, University of Perugia, Italy*

Abstract. The large development of wireless services and the scarcity of the usable frequencies require an efficient use of the radio spectrum which guarantees interference avoidance. The Channel Assignment (CA) problem achieves this goal by partitioning the radio spectrum into disjoint channels, and assigning channels to the network base stations so as to avoid interference. On a flat region without geographical barriers and with uniform traffic load, the network base stations are often placed according to a regular plane tessellation, while the channels are permanently assigned to the base stations. This paper surveys the CA problem on grid network topologies, where the plane is tessellated by regular polygons. Interference between two base stations at a given distance is avoided by forcing the channels assigned to such stations to be separated by a gap which is proportional to the distance between the stations. Under these assumptions, the CA problem can be modeled as a suitable coloring problem. Formally, given an undirected graph $G = (V, E)$ and a vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ of positive integers, an $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of G is a function f from the vertex set V to a set of nonnegative integers such that $|f(u) - f(v)| \geq \delta_i$, if $d(u, v) = i,\ 1 \leq i \leq \sigma - 1,$ where $d(u, v)$ is the distance (i.e. the minimum number of edges) between the vertices u and v. An optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring for G is one minimizing the largest used integer over all such colorings. This paper surveys efficient algorithms for finding optimal $L(2, 1)$- and $L(2, 1, 1)$-colorings of honeycomb, square, and cellular grids.

Keywords. Wireless Networks, Channel Assignment, Interferences, Honeycomb Grids, Square Grids, Cellular Grids, $L(2, 1)$-coloring, $L(2, 1, 1)$-coloring

Introduction

In a wireless network, the main difficulty against an efficient use of the radio spectrum is given by interferences, caused by unconstrained simultaneous transmissions, which result in damaged communications. The Channel Assignment (CA) problem is the task of efficiently assigning the radio spectrum to the set of base stations of the network. Such a problem, that first appeared in TV broadcasting and military communications in late 1960s, keeps renewing its interest due to the large development of wireless telephone

[1]Corresponding Author: Cristina M. Pinotti, Department of Computer Science and Mathematics, University of Perugia, Via Vanvitelli 1, 06123 Perugia, Italy; E-mail: pinotti@unipg.it

networks (e.g. FDMA, TDMA, GSM networks) and satellite communication [1]. Although there are many different models, all scenarios are characterized by a set of transmitters (usually, antennae), a set of disjoint channels (frequencies) obtained partitioning the radio spectrum, and a strategy for assigning channels to transmitters so that data communications are possible.

The channel assignment can be done following several strategies [20]. In the Fixed Channel Assignment (FCA), channels are statically assigned to the transmitters for their exclusive and permanent use, and remain stable over time [1,9,23]. Opposite to FCA, Dynamic Channel Assignment (DCA) maintains all channels in a central pool, and dynamically assigns them to the transmitters for temporary use [11,15,16]. Finally, Hybrid Channel Assignment (HCA) combines the two above strategies [20]. FCA performs well when the traffic load is uniform in time and in space, because it yields maximum channel reusability. In contrast, DCA is more suited in the case of short-term temporal and spatial traffic variations, because it privileges the flexibility of the channel allocation with respect to the channel reusability. Under mixed traffic conditions, either HCA or FCA with borrowing are used. HCA, which has channels partitioned into fixed and dynamic sets, performs well when on the top of a constant traffic load there is a fraction of highly variable communications. In the FCA with borrowing, a transmitter which has used all its statically assigned channels can occasionally borrow free channels from its neighbouring transmitters.

This paper concentrates on FCA. Using this technique, CA can be modeled as variants of vertex graph coloring [1,19,24]. Formally, an undirected graph $G = (V, E)$ models the wireless network, where the vertices in V represent the transmitters and the edges in E represent pairs of transmitters that may potentially interfere. The separation required to avoid interference between the frequencies assigned to the edge end-points is represented by a label of the edge. Colors (i.e. frequencies) have to be assigned to the vertices so that the separation constraints are verified and an objective function is optimized. Typical objective functions range from minimizing the difference between the largest and the lowest used colors, while avoiding interferences (called, Minimum-Span), up to minimizing interferences using a given number of colors (called, Fixed-Spectrum).

For arbitrary network topologies and general separation constraints, the resulting vertex coloring problems are computationally intractable (i.e., NP-hard). Therefore, the FCA problem is usually addressed by means of heuristic approaches, like genetic algorithms, taboo search, saturation degree, simulated annealing, and ants heuristics, just to name a few [1]. The performance of such heuristics is compared on widely accepted benchmarks, like CELAR data, COST 259 data, and Philadelphia instances. In particular, the Philadelphia instances, that have been heuristically solved to optimal for the Minimum-Span objective function, suggest the relevance of topologies based on regular tessellations of the plane. In such a case, the interference phenomena depend on the distance among the antennae. Thus, the separation constraints are modeled by a separation vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ of positive integers such that channels assigned to base stations at distance i be at least δ_i apart [1,19,20], which implies that the same color can be reused only at stations whose distance is at least σ. Typical values of the *co-channel reuse distance* σ studied so far are upper bounded by 5, while typical values of the separations are $\delta_1 = 3$ or 2, $\delta_2 = 2$ or 1, and $\delta_3 = \delta_4 = 1$ [1]. However, in the next generation of wireless access systems, due to the decreasing cost of infrastructures and to the need of wider bandwidth, a large number of small cells, each with significant power, is expected

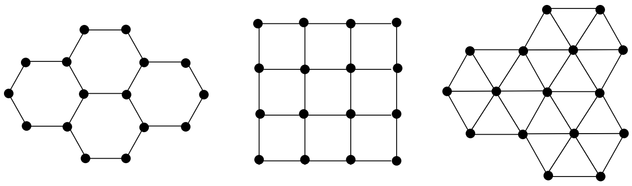

Figure 1. Possible grids of 16 vertices: (a) honeycomb grid, (b) square grid, and (c) cellular grid.

to cover a wider communication region [31]. Therefore, the upper bound of σ is expected to become larger than 5.

In this paper networks modeled by regular plane tessellations are considered. It is well-known that there are only three different tessellations of the plane which use regular polygons, namely hexagons, squares, and triangles. Such tessellations can be used to place at the polygon vertices the base stations of the wireless communication networks, leading to three well-known topologies: *honeycomb*, *square*, and *cellular* grids, depicted in Fig. 1 for 16 vertices. Each of these grids has its own pros and cons. The cellular networks are currently the most important to the radio engineer because the transmission areas of their stations cover the whole plane using the smallest possible transmitter density. However, since the required transmitter power increases linearly with the bandwidth, high speed radio access can be guaranteed in dense ubiquitous infrastructures, like cellular networks, only at tremendous costs. As a possible alternative to dense infrastructures, sparse infrastructures of information-kiosks, called infostations, close to which high data rate communication is possible, have been introduced. Examples of such infrastructures distributed in a Manhattan fashion, modeled by square grids, are already available inside big cities [31]. Moreover, the performance of a topology can be evaluated with respect to several parameters, such as *degree*, *diameter*, and *cost*, which is defined as the product of the degree and diameter. Comparing the above three grids in terms of such parameters, measured with respect to the same number of vertices, one notes that a honeycomb grid has the smallest degree and cost, a cellular grid has the smallest diameter, while a square grid is always worse than at least one of the other two grids [28].

Summarizing, this paper reviews the Minimum-Span FCA problem on grid network topologies, under the assumption that a single channel has to be assigned to each station, and surveys several algorithms which find optimal solutions in polynomial time [9,26, 27]. It is worth noting that such solutions can be used to derive sub-optimal solutions in the more general uniform multi-coloring case, where the same number m of channels has to be assigned to each vertex. Indeed, this can be accomplished by optimally assigning one color per vertex, e.g. using s colors in total, and then coloring each vertex by m colors repeatedly shifted s channels up. Precisely, if a vertex gets the single color i, then it receives also colors $i+s, i+2s, \ldots, i+(m-1)s$.

In this paper, the FCA problem is modeled as follows. Let $G=(V,E)$ be an undirected graph representing the network topology and let $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ be the separation vector, with $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{\sigma-1}$. A k-$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring of G is a function f from the vertex set V to the set of nonnegative integers $\{0, \ldots, k\}$ such that $|f(u) - f(v)| \geq \delta_i$, if $d(u,v) = i,\ 1 \leq i \leq \sigma - 1,$ where $d(u,v)$ is the distance (i.e. the minimum number of edges) between the vertices u and v. An *optimal*

$L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring for G is one minimizing k over all such colorings. The largest color used by an optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring is denoted by $\lambda(G)$. Note that, since the set of colors includes 0, the overall number of colors involved by an optimal coloring f is in fact $\lambda(G) + 1$ (although, due to the channel separation constraint, some colors in $\{1, \ldots, \lambda(G)-1\}$ might not be actually assigned to any vertex). Thus, the channel assignment problem consists of finding an optimal $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring for G. Note that an $L(1)$-coloring is just a classical vertex coloring and in this case $\lambda(G) + 1 = \chi(G)$, the chromatic number of G.

The channel assignment problem has been widely studied when the separation vector $(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$ is equal to $(1, 1, \ldots, 1)$ [3,5,14,22,26]. In particular, the intractability of optimal $L(1, 1, \ldots, 1)$-coloring, for any positive integer σ, has been proved by McCormick [22]. In contrast, optimal $L(1, 1, \ldots, 1)$-colorings have been proposed in [5,7] for rings, square grids, and honeycomb grids, and in [2] for trees and interval graphs. Moreover, when the separation vector is $(\delta_1, 1, \ldots, 1)$, optimal $L(\delta_1, 1, \ldots, 1)$-colorings have been proposed in [9,27] for rings, square grids, and cellular grids. Optimal solutions have been proposed for the $L(\delta_1, \delta_2)$-coloring problem on rings [17] and on square and cellular grids [30]. This latter paper provided also an optimal $L(2, 1, 1)$-coloring for square grids. The $L(2, 1, 1)$-coloring problem has been also optimally solved for cellular grids, honeycomb grids, and rings in [8,9]. The $L(2, 1)$-coloring has been investigated in [8,10,13,18,25]. Bodlaender et al. [10] proved that the $L(2, 1)$-coloring problem is NP-hard for planar, bipartite, and chordal graphs, and presented approximate solutions for outerplanar, permutation and split graphs. Moreover, $L(2, 1)$-colorings for unit interval graphs and trees have been found, respectively, by Sakai [25] and by Chang and Kuo [13]. A recent annotated bibliography on the $L(\delta_1, \delta_2)$-coloring problem for several special classes of graphs can be found in [12]. As a related case, when $(\delta_1, \delta_2) = (0, 1)$, the $L(0, 1)$-coloring problem models that of avoiding only the so-called *hidden interferences*, due to stations which are outside the hearing range of each other and transmit to the same receiving station. Optimal $L(0, 1)$-colorings have been provided for square grids in [21], whereas the intractability of optimal $L(0, 1)$-coloring has been proved in [4], where also optimal solutions for rings and complete binary trees are given. For arbitrary graphs and general separation vectors, the $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring problem is faced by heuristics [1]. However, approximation algorithms have been discussed in [6] for trees and interval graphs.

This paper surveys efficient algorithms which find optimal solutions for the $L(2, 1)$- and $L(2, 1, 1)$-coloring problems on honeycomb, square, and cellular grids. The graph theoretical approach outlined in [14,18,21,22,26] is followed and the results proposed in [7,8,9,30] are surveyed. The rest of the present paper is structured as follows. Section 1 briefly recalls some preliminary graph theoretical results (e.g., augmented graph, clique) that will be used in the following sections. Moreover, a simple distributed scheme is sketched to allow the vertices to compute their own relative positions in the grid, in case these info are not already available. Such positions will then be used by the vertices to self-assign their proper channel in constant time. Sections 2, 3, and 4 provide simple algorithms based on periodic and arithmetic rules to optimally solve the $L(2, 1)$- and $L(2, 1, 1)$-coloring problems on honeycomb, square, and cellular grids, respectively. Finally, conclusions are offered in Section 5, where it is also pointed out when the channel assignment solutions proposed in this paper are better than those obtained by employing guard frequencies between adjacent channels.

1. Preliminaries

The channel assignment problem on a network N with no channel separation constraint and co-channel reuse distance σ, namely the $L(1, 1, \ldots, 1)$-coloring problem, can be reduced to a classical coloring problem on an *augmented* graph $G_{N,\sigma}$, which is the $(\sigma-1)$-th power of G and is obtained as follows. The vertex set of $G_{N,\sigma}$ is the same as the vertex set of N, while an edge $[r, s]$ belongs to the edge set of $G_{N,\sigma}$ if and only if the distance $d(r, s)$ between the vertices r and s in N satisfies $d(r, s) \leq \sigma - 1$. Now, colors must be assigned to the vertices of $G_{N,\sigma}$ so that every pair of vertices connected by an edge is assigned a couple of different colors and the minimum number of colors is used. Hence, the role of *maximum cliques* in $G_{N,\sigma}$ is apparent for deriving lower bounds on the minimum number of channels for the $L(1, 1, \ldots, 1)$-coloring problem on N. A *clique* for $G_{N,\sigma}$ is a subset of vertices of $G_{N,\sigma}$ such that there is an edge for each pair of vertices in the subset. A clique of size n in the augmented graph $G_{N,\sigma}$ implies that at least n different colors are needed to color $G_{N,\sigma}$. In other words, the size of the largest clique in $G_{N,\sigma}$ is a lower bound for the number of channels required to solve the channel assignment problem without channel separation constraint. Clearly, in the presence of both channel separation and co-channel reuse distance constraints, at least as many channels are required as in the presence of the channel separation constraint only. Formally, a lower bound for the $L(1, 1, \ldots, 1)$-coloring problem is also a lower bound for the $L(\delta_1, 1, \ldots, 1)$-coloring problem, with $\delta_1 \geq 1$. In particular, lower bounds for $L(1, 1)$- and $L(1, 1, 1)$-colorings hold also for $L(2, 1)$- and $L(2, 1, 1)$-colorings, respectively.

Let the *complement graph* $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ be the graph having the same vertex set V as G and having the edge set $\overline{E}$ obtained by swapping edges with non-edges in E. Recall that a *Hamilton path* is a path that traverses each vertex of a graph exactly once. The following two lemmas are due to Griggs and Yeh [18].

Lemma 1. *Consider the $L(2, 1, \ldots, 1)$-coloring problem on a graph $G = (V, E)$ such that $d(u, v) < \sigma$ for every pair of vertices u and v in V. Then, $\lambda(G) = |V| - 1$ if and only if $\overline{G}$ has a Hamilton path.*

Proof. To satisfy the channel separation constraint, two vertices of G may get two consecutive colors if and only if they are not adjacent, that is, if and only if they are adjacent in $\overline{G}$. If $\lambda(G) = |V| - 1$ then there is an ordering $v_0, v_1, \ldots, v_{|V|-1}$ of the vertices such that $f(v_i) = i$, for $i = 0, 1, \ldots, |V| - 1$. Therefore, $v_0, v_1, \ldots, v_{|V|-1}$ is a Hamilton path for $\overline{G}$. Conversely, if $\overline{G}$ has a Hamilton path $v_0, v_1, \ldots, v_{|V|-1}$, then the optimal $L(\delta_1, 1, \ldots, 1)$-coloring is $f(v_i) = i$, for $i = 0, 1, \ldots, |V| - 1$. $\square$

Consider the *star* graph S_Δ which consists of a *center* vertex c with degree Δ, and Δ *ray* vertices of degree 1.

Lemma 2. *Let the center c of S_Δ be already colored. Then, the largest color required for a k-$L(2, 1)$-coloring of S_Δ is at least:*

$$k = \begin{cases} \Delta + 1 & \textit{if } f(c) = 0 \textit{ or } f(c) = \Delta + 1 \\ \Delta + 2 & \textit{if } 0 < f(c) < \Delta + 1 \end{cases}$$

Proof. By Lemma 1, $k \geq \Delta + 1$ because $\overline{S}_\Delta$ has no Hamilton path. If $f(c) = 0$ or $f(c) = \Delta + 1$, either color 1 or Δ violates the channel separation constraint and cannot be used at the ray vertices. Similarly, if $0 < f(c) < \Delta$, both colors $f(c)-1$ and $f(c)+1$ cannot be used at the ray vertices. Therefore, one or two extra colors are required with respect to the star size, depending on the center color $f(c)$. □

The channel assignment algorithms to be presented allow any vertex to self-assign its proper channel in constant time, provided that it knows its relative position within the network. If this is not the case, such relative positions can be computed for all the vertices using simple distributed algorithms requiring optimal time and optimal number of messages. Assume that each vertex of the network only knows its own geographic position (e.g. by means of its I.D. or a local geographic position system (GPS)) and the names of its neighbours (which can be easily obtained by the usual topology-exchange distributed algorithm [29]). The vertices are assumed to be asynchronous and can communicate by exchanging control messages (e.g. via dedicated system signals such as SS7 or MAC protocols such as ALOHA). There is only one kind of control message, which is sent by a vertex to tell its geographic position and its relative position to its neighbours. The computation is started by a single vertex, which is the only vertex which initially knows its relative position. When a vertex receives a control message from a neighbour, it is capable of recognizing whether the sender is a North, South, East, or West neighbour, by comparing its geographic position and that of the sender (the agreement about the actual cardinality points can be established and broadcast by the vertex starting the computation, after knowing the GPS positions of its neighbors). When a vertex receives a control message from a neighbour, if it has not yet computed its position and some conditions are met, then it computes its own relative position and in turn sends a control message, otherwise it neglects the message.

2. Honeycomb Grids

A *honeycomb grid* H of size $r \times c$ has r rows and c columns, indexed respectively from 0 to $r-1$ (from top to bottom) and from 0 to $c-1$ (from left to right), with $r \geq 3$ and $c \geq 2$. A generic vertex u of H will be denoted by $u = (i, j)$, where i is its row index and j is its column index. Each vertex has degree 3, except for some vertices on the borders. In particular, each vertex (i, j), which does not belong to the grid borders, is adjacent to the following 3 vertices:

1. $\begin{cases} (i, j+1) \text{ if } (i \text{ is even and } j \text{ is even}) \text{ or } (i \text{ is odd and } j \text{ is odd}) \\ \text{or} \\ (i, j-1) \text{ if } (i \text{ is even and } j \text{ is odd}) \text{ or } (i \text{ is odd and } j \text{ is even}) \end{cases}$
2. $(i-1, j)$
3. $(i+1, j)$

The optimal solutions for the $L(2,1)$- and $L(2,1,1)$-coloring problems on honeycomb grids surveyed in this section have been provided in [7,8].

Lemma 3. *For $r \geq 3$ and $c \geq 3$ there is a k-$L(2,1)$-coloring of a honeycomb grid H of size $r \times c$ only if $k \geq 5$.*

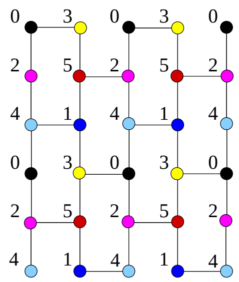

Figure 2. Optimal coloring obtained by the Honeycomb-5-$L(2,1)$-coloring algorithm.

Proof. It follows immediately from Lemma 2 because there is at least one vertex of degree 3 that cannot be colored either 0 or 4. Hence, $\lambda(H) \geq 5$. ☐

Below, an optimal 5-$L(2,1)$-coloring is given to color all the vertices of any honeycomb grid H of size $r \times c$, with $r \geq 3$ and $c \geq 3$.

Algorithm Honeycomb-5-$L(2,1)$**-coloring** (H, r, c);

- If $r \geq 3$ and $c \geq 3$, then assign to each vertex $u = (i, j)$ the color

$$f(u) = (2i + 3j) \bmod 6$$

An optimal coloring for a honeycomb grid of size 6×5 is illustrated in Figure 2.

Theorem 1. *The Honeycomb-5-$L(2,1)$-coloring algorithm is optimal for any honeycomb grid H of size $r \times c$, with $r \geq 3$ and $c \geq 3$.*

Proof. To prove the theorem, it must be shown that the algorithm satisfies the channel separation and co-channel reuse constraints, and that it uses the minimum number of colors.

Consider a generic vertex $u = (i, j)$ of H. The channel separation constraint is easily verified for any vertex v adjacent to u. Indeed, the color $f(v)$ can be rewritten in terms of $f(u)$ as follows:

$$f(v) = \begin{cases} (f(u) + 3) \bmod 6 & \text{if } [v = (i, j+1),\ i \text{ is even, and } j \text{ is even}] \\ & \text{or } [v = (i, j+1), i \text{ odd, and } j \text{ is odd}] \\ (f(u) - 3) \bmod 6 & \text{if } [v = (i, j-1),\ i \text{ is even, and } j \text{ is odd}] \\ & \text{or } [v = (i, j-1),\ i \text{ is odd, and } j \text{ is even}] \\ (f(u) - 2) \bmod 6 & \text{if } v = (i-1, j) \\ (f(u) + 2) \bmod 6 & \text{if } v = (i+1, j) \end{cases}$$

The above coloring assigns the three different colors $(f(u) - 2) \bmod 6$, $(f(u) + 2) \bmod 6$, and $(f(u) + 3) \bmod 6$ to the three vertices adjacent to u. Thus, since any two vertices at distance 2 are both adjacent to a common vertex u, vertices at distance 2 verify the co-channel reuse constraint. The coloring optimality follows from Lemma 3. ☐

Lemma 4. *For $r \geq 4$ and $c \geq 3$, or $r \geq 5$ and $c = 2$, there is a k-$L(2,1,1)$-coloring of a honeycomb grid H only if $k \geq 6$.*

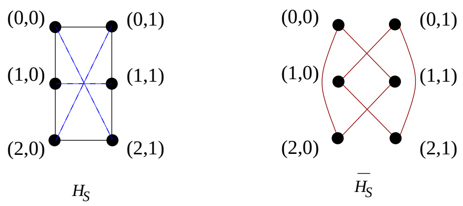

Figure 3. The subgraph H_S with the dummy edges (dashed), and its complement $\overline{H}_S$.

Proof. Consider the augmented graph $G_{H,4} = (V, E')$ and the subset of vertices $S = \{(0,0),(0,1),\ (1,0),(1,1),(2,0),(2,1)\}$ Since all the 6 vertices in S are mutually at distance at most 3 in H, they form a clique in $G_{H,4}$. Therefore, $\lambda(H) \geq 5$.

In the case where $r \geq 4$ and $c \geq 3$, consider the subgraph H_S induced by S and also the vertex $(3,0)$. To satisfy the co-channel reuse constraint using exactly 6 colors, vertex $(3,0)$ must get the same color as vertex $(0,1)$. Moreover, due to the channel separation constraint, the colors assigned to vertices $(2,0)$ and $(3,0)$ must have a gap of at least $\delta_1 = 2$. Hence, also the colors of $(2,0)$ and $(0,1)$ must have a gap of at least $\delta_1 = 2$. This is equivalent to add in H_S a *dummy* edge between vertices $(2,0)$ and $(0,1)$. The same reasoning can be repeated for the pairs of vertices $(1,1)$ and $(1,0)$, and $(2,1)$ and $(0,0)$, as illustrated in Figure 3. Now, consider the complement $\overline{H}_S$ of H_S, depicted also in Figure 3. There is no Hamilton path in $\overline{H}_S$ because it consists of two connected components. It follows from Lemma 1 that $\lambda(H) \geq 6$.

By a similar argument, the same lower bound can be proved for honeycomb grids when $r \geq 5$ and $c = 2$. Indeed, when $r \geq 5$ and $c = 2$, vertices $(2,0)$ and $(2,1)$ belong to two distinct cliques in $G_{H,4}$. Then, to keep $\lambda(H) = 5$, the same colors used for vertices $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ must be reused for vertices $(3,0)$, $(3,1)$, $(4,0)$, $(4,1)$. In particular, as explained before, vertices $(3,0)$ and $(3,1)$ must get the same colors as $(0,1)$ and $(0,0)$. Thus, vertices $(4,1)$ and $(4,0)$ must get the same colors as $(1,0)$ and $(1,1)$. As said, this is equivalent to add in H_S the three *dummy* edges between the pairs of vertices $(2,0)$ and $(0,1)$, $(2,1)$ and $(0,0)$, and $(1,0)$ and $(1,1)$. As before, there is no Hamilton path in $\overline{H}_S$. □

Below, an optimal 6-$L(2,1,1)$-coloring is given to color all the vertices of a honeycomb grid H of size $r \times c$, with $r \geq 4$ and $c \geq 3$ or $r \geq 5$ and $c = 2$.

Algorithm Honeycomb-6-$L(2,1,1)$-coloring (H, r, c);

- If $r \geq 4$ and $c \geq 3$ or $r \geq 5$ and $c = 2$, assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } (i \equiv 0 \bmod 6 \text{ and } j \text{ is even}) \text{ or } (i \equiv 3 \bmod 6 \text{ and } j \text{ is odd}) \\ 4 & \text{if } (i \equiv 0 \bmod 6 \text{ and } j \text{ is odd}) \text{ or } (i \equiv 3 \bmod 6 \text{ and } j \text{ is even}) \\ 6 & \text{if } (i \equiv 1 \bmod 6 \text{ and } j \text{ is even}) \text{ or } (i \equiv 4 \bmod 6 \text{ and } j \text{ is odd}) \\ 2 & \text{if } (i \equiv 1 \bmod 6 \text{ and } j \text{ is odd}) \text{ or } (i \equiv 4 \bmod 6 \text{ and } j \text{ is even}) \\ 1 & \text{if } (i \equiv 2 \bmod 6 \text{ and } j \text{ is even}) \text{ or } (i \equiv 5 \bmod 6 \text{ and } j \text{ is odd}) \\ 5 & \text{if } (i \equiv 2 \bmod 6 \text{ and } j \text{ is odd}) \text{ or } (i \equiv 5 \bmod 6 \text{ and } j \text{ is even}) \end{cases}$$

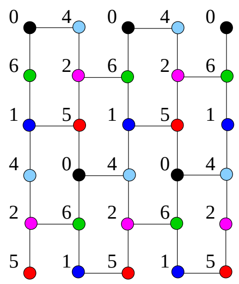

Figure 4. Optimal coloring obtained by the Honeycomb-6-$L(2, 1, 1)$-coloring algorithm.

The optimal $L(2, 1, 1)$-coloring for the honeycomb grid depicted in Figure 2 is illustrated in Figure 4.

Theorem 2. *The Honeycomb-6-$L(2, 1, 1)$-coloring algorithm is optimal for honeycomb grids of size $r \times c$, with $r \geq 4$ and $c \geq 3$ or with $r \geq 5$ and $c = 2$.*

Proof. Consider a generic vertex $u = (i, j)$ of H. By construction, the channel separation constraint is immediately verified. Indeed, for any vertex v adjacent to u such that $v = (i, j \pm 1)$, $f(v) = f(u) \pm 4$ holds. Moreover, as shown in Figure 4, any pair u, v of adjacent vertices on the same column can be colored only as follows $f(u) = 0$ and $f(v) = 6$, $f(u) = 6$ and $f(v) = 1$, $f(u) = 1$ and $f(v) = 4$, $f(u) = 4$ and $f(v) = 2$, $f(u) = 2$ and $f(v) = 5$, $f(u) = 5$ and $f(v) = 0$. Therefore, the gap between the colors assigned to each pair of adjacent vertices is at least 2.

Now, in order to prove that the co-channel reuse constraint is verified, let us show that two vertices colored the same are at distance greater than 3. First of all, note that each row of H is colored with two colors, and any 3 consecutive rows of H use different colors. The i-th and $(i+3)$-th rows of H, for any i, are colored with the same two colors, except that the order of such two colors along the row is swapped. That is, vertices (i, j) and $(i, j + 1)$ are colored, respectively, as vertices $(i + 3, j + 1)$ and $(i + 3, j)$. Hence, two vertices in rows i and $(i + 3)$ get the same color only if their distance is at least 4. Moreover, the i-th and $(i + 6)$-th rows of H, for any i, are colored the same. Hence, the same color can be reused on the same column only in two vertices at distance 6. Finally, all the even (resp., odd) columns are colored the same. Although the vertices (i, j) and $(i, j + 2)$ get the same color, their distance is 4 because there are no two consecutive horizontal edges. Summarizing, a given color is reused on H according to the pattern shown in Figure 5. The coloring optimality follows from Lemma 4. $\square$

2.1. Special Cases

Note that the honeycomb grid was defined with $r \geq 3$ because for $r = 1$ it reduces to a set of isolated edges plus a possible isolated vertex, while for $r = 2$ or $c = 1$ it consists of a bus, namely a simple path, for which both the $L(2, 1)$- and $L(2, 1, 1)$-colorings can be easily solved having $\lambda(H) = 4$ as the largest used color [8,13]. Therefore, the only special cases left out are the following:

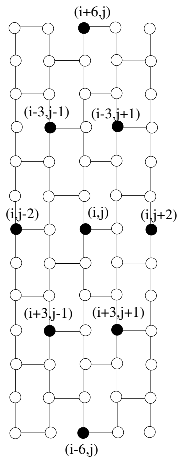

Figure 5. The vertices closest to vertex $u = (i, j)$ where the color $f(u)$ is reused.

Lemma 5. *An optimal $L(2,1)$-coloring of a honeycomb grid H has*

$$\lambda(H) = \begin{cases} 4 & \textit{if } 3 \leq r \leq 4 \textit{ and } c = 2 \\ 5 & \textit{if } r \geq 5 \textit{ and } c = 2 \end{cases}$$

Proof. When $3 \leq r \leq 4$ and $c = 2$, H contains a ring R of size $n = 6$, which has a lower bound on $\lambda(R)$ of 4 [18]. A 4-$L(2,1)$-coloring for H of size 4×2 is: $f(0,0) = f(2,1) = 0, f(1,0) = f(1,1) = 2, f(2,0) = f(0,1) = 4, f(3,0) = 1, f(3,1) = 3$.

When $r = 5$ and $c = 2$, H consists of two rings of size $n = 6$, which share an edge e. Assuming $\lambda(H) = 4$, by [18], each ring should be colored using only the colors $0, 2$ and 4. By Lemma 2, there is a 4-$L(2,1)$-coloring for H only if the endpoints of e get the colors 0 and 4, which forces the other 4 vertices adjacent to the endpoints of e to be colored only by 2. This violates the co-channel reuse constraint, because some of these 4 vertices are at distance 2. Therefore, $\lambda(H) \geq 5$. In this case, a 5-$L(2,1)$-coloring is obtained by Algorithm Small-Honeycomb-$L(2,1)$-coloring. □

Algorithm Small-Honeycomb-$L(2,1)$-coloring (H, r, c);

- If $r \geq 5$ and $c = 2$, assign to each vertex $u = (i, j)$ the color

$$f(u) = (2i + 3j) \bmod 6$$

Lemma 6. *When $3 \leq r \leq 4$ and $c = 2$ or $r = 3$ and $c \geq 3$, an optimal $L(2,1,1)$-coloring of a honeycomb grid H has $\lambda(H) = 5$.*

Proof. Consider a honeycomb grid H of size 3×2. Since the augmented graph $G_{H,4}$ is a clique, $\lambda(H) \geq 5$. An optimal 5-$L(2,1,1)$-coloring for H of size 4×2 is: $f(0,0) = f(3,1) = 0, f(0,1) = f(3,0) = 4, f(1,0) = 3, f(1,1) = 2, f(2,0) = 1, f(2,1) = 5$. An optimal 5-$L(2,1,1)$-coloring for H when $r = 3$ and $c \geq 3$ is obtained by Algorithm Small-Honeycomb-$L(2,1,1)$-coloring. □

Algorithm Small-Honeycomb-$L(2,1,1)$-coloring (H,r,c);

- If $r=3$ and $c \geq 3$, assign to each vertex $u=(i,j)$ the color

$$f(u) = \begin{cases} 3 & \text{if } r=0 \text{ and } c \equiv 0 \bmod 2 \\ 0 & \text{if } r=0 \text{ and } c \equiv 1 \bmod 2 \\ 1 & \text{if } r=1 \text{ and } c \equiv 0 \bmod 2 \\ 4 & \text{if } r=1 \text{ and } c \equiv 1 \bmod 2 \\ 5 & \text{if } r=2 \text{ and } c \equiv 0 \bmod 2 \\ 2 & \text{if } r=2 \text{ and } c \equiv 1 \bmod 2 \end{cases}$$

2.2. Distributed Computation of Relative Positions

The channel assignment algorithms presented above allow any vertex to self-assign its proper channel in constant time provided that it knows its relative position within the honeycomb grid. If this is not the case, such relative positions can be computed for all the vertices using a simple distributed algorithm requiring optimal time and optimal number of messages, as detailed below.

The computation is started by the upper-left corner vertex in the honeycomb grid, which is the only vertex knowing its position $(0,0)$. A control message is structured as $CM(v, g_v, i, j)$, where g_v and (i,j) are the geographic and relative positions of v, respectively. When a vertex u receives $CM(v, g_v, i, j)$ from a North neighbour v and $i \geq 1$, then u computes its relative position $(i+1, j)$ and sends $CM(u, g_u, i+1, j)$ so as to propagate the computation downwards along the columns of the honeycomb grid. In the first two rows, however, different conditions have to be dealt with. Specifically, if v is a West neighbour of u and $i=0$, then u computes its position $(0, j+1)$ and sends $CM(u, g_u, 0, j+1)$, while if v is a South neighbour of u and $i=1$, then u computes $(0,j)$ and sends $CM(u, g_u, 0, j)$. As for the vertices in the second row, if v is a North neighbour and $i=0$, then u computes $(1,j)$ and sends $CM(u, g_u, 1, j)$, while if v is a West neighbour and $i=1$, then u computes $(1, j+1)$ and sends $CM(u, g_u, 1, j+1)$.

It is easy to see that the overall number of messages required is $O(rc)$ while the total time is $O(r+c)$, assuming that a message reaches its destination in $O(1)$ time. Since there are rc vertices in the grid and the grid diameter is $O(r+c)$, the channel assignment for all the vertices can be performed in a distributed fashion so as to require an optimal time and an optimal number of messages.

3. Square Grids

A *square grid* B of size $r \times c$, with both $r \geq 2$ and $c \geq 2$, can be obtained from a honeycomb grid H of the same size connecting all the pairs of consecutive nodes lying on the same row. Thus, a generic vertex $u=(i,j)$ of B, which does not belong on the borders, has degree 4. In particular, vertex $u=(i,j)$ is adjacent to the vertices $(i-1,j), (i,j+1), (i+1,j)$ and $(i,j-1)$.

The optimal solutions for the $L(2,1)$- and $L(2,1,1)$-coloring problems on square grids surveyed in this section have been proposed in [8,30].

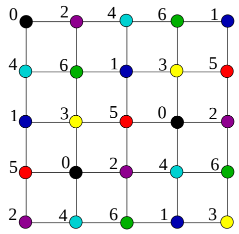

Figure 6. Optimal coloring obtained by the Grid-6-$L(2,1)$-coloring algorithm.

Lemma 7. *There is a k-$L(2,1)$-coloring of a square grid B of size $r \times c$, with $r \geq 3$ and $c \geq 3$, only if $k \geq 6$.*

Proof. The lower bound for k follows immediately because there is at least a vertex of B with degree 4 that cannot be colored either 0 or 5. Hence, from Lemma 2, $\lambda(B) = 6$. $\square$

In the following, an algorithm for optimally $L(2,1)$-coloring a square grid B of size at least 3×3 is given.

Algorithm Grid-6-$L(2,1)$-coloring (B, r, c);

- If $r \geq 3$ and $c \geq 3$, assign to each vertex $u = (i,j)$ the color

$$f(u) = (2i + 4j) \bmod 7$$

Theorem 3. *The Grid-6-$L(2,1)$-coloring algorithm is optimal.*

Proof. When $r \geq 3$ and $c \geq 3$, consider a generic vertex $u = (i,j)$ of B. The channel separation constraint is easily verified for any vertex v adjacent to $u = (i,j)$ because

$$f(v) = \begin{cases} (f(u) \pm 2) \bmod 7 & \text{if } v = (i, j \pm 1) \\ (f(u) \pm 3) \bmod 7 & \text{if } v = (i \pm 1, j) \end{cases}$$

Moreover, the co-channel reuse constraint is verified because the 4 vertices closest to u and colored as u are $(i+1, j-2)$, $(i-1, j+2)$, $(i-2, j-3)$, $(i+2, j+3)$, as can be easily checked observing Figure 6. Finally, the optimality follows from Lemma 7. $\square$

Lemma 8. *There is a k-$L(2,1,1)$-coloring of a square grid B, with $r \geq 5$ and $c \geq 4$ or $r \geq 4$ and $c \geq 5$, only if $k \geq 8$.*

Proof. For a square grid $B = (V, E)$ of size $r \times c$, with $r \geq 5$ and $c \geq 4$, consider the augmented graph $G_{B,4} = (V, E')$. For any pair of vertices on the same column $u = (i,j)$ and $v = (i+3, j)$, with $0 \leq i \leq r-4$ and $1 \leq j \leq c-2$, let $S_{u,v}$ be the subset of vertices $\{(i,j), (i+1,j), (i+2,j), (i+3,j), (i+1,j-1), (i+2,j-1), (i+1,j+1), (i+2,j+1)\}$ at pairwise distance no more than 3. Similarly, for any pair of vertices on the same row $u = (i,j)$ and $w = (i, j+3)$, with $1 \leq i \leq r-2$ and $0 \leq j \leq c-4$, let $S'_{u,w}$ be the subset of vertices $\{(i,j), (i,j+1), (i,j+2), (i,j+3), (i+1,j+1), (i+1,j+2), (i-1,j+1), (i-1,j+2)\}$ at pairwise distance no more than 3. Since both $S_{u,v}$ and

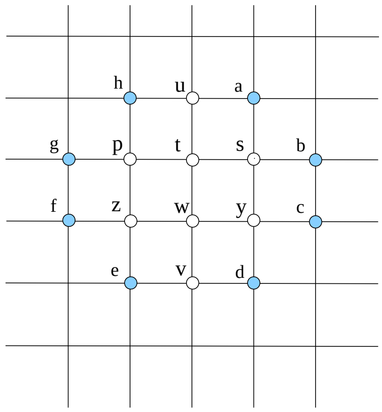

Figure 7. The subsets of vertices $L_{u,v}$ (all) and $S_{u,v}$ (white).

$S'_{u,w}$ induce a clique in $G_{B,4}$, at least 8 colors are needed to satisfy the co-channel reuse constraint. However, as proved in the following, 8 colors are not enough to color the set $L_{u,v}$ of vertices depicted in Figure 7, which consists of $S_{u,v}$ along with all the vertices of B at horizontal distance exactly 1 from the vertices on the border of $S_{u,v}$.

Indeed, to color vertices $a = (i, j + 1)$ and $b = (i + 1, j + 2)$, consider the vertex $p = (i+1, j-1)$, the subsets of vertices $S_{u,v}$ and $S'_{p,b}$, and the square subgrid M induced by $S_{u,v}$. Once $S_{u,v}$ has been assigned to all different colors, the two vertices b and a of $S'_{p,b}$ must be assigned to the two colors used for the two vertices $z = (i + 2, j - 1)$ and $v = (i + 3, j)$ of $S_{u,v}$, if only 8 colors are to be used. Due to the channel separation constraint, the colors assigned to vertices a and b must be at least 2 apart from the color assigned to the vertex $s = (i + 1, j + 1)$. This is equivalent to add to M two dummy edges: one between vertices s and z, and the other between vertices s and v, as shown in Figure 8. Now repeating the same argument for the three pairs of vertices $c = (i+2, j+2)$ and $d = (i+3, j+1)$, $e = (i+3, j-1)$ and $f = (i+2, j-2)$, and $g = (i+1, j-2)$ and $h = (i, j - 1)$, other dummy edges must be added (see Figure 9), namely those between p and y, u and y, u and z, s and z, p and y, and p and v.

By the previous discussion, either vertex h or g is colored as vertex v. Analogously, either vertex f or e is colored as vertex u. Examining the set of vertices $\{v, e, f, g, h, u\}$, it is easy to be convinced that whatever is the color assignment adopted for such vertices, the colors $f(u)$ and $f(v)$ must be assigned to two adjacent vertices among $\{v, e, f, g, h, u\}$. Namely, $f(u)$ and $f(v)$ must appear in one of the following pair of vertices: u and h, or v and e, or f and g. Thus, one further dummy edge between vertices u and v must be added to M, as shown in Figure 9.

Finally, let us build $\overline{M}$, the complement of M. Since $\overline{M}$ consists of two connected components (see also Figure 9), M does not contain a Hamilton path. Hence, by Lemma 1, there is no 7-$L(2, 1, 1)$-coloring for square grids of size $r \times c$, with $r \geq 5$ and $c \geq 4$. Thus, $\lambda(B) \geq 8$. The proof when $r \geq 4$ and $c \geq 5$ is analogous. □

The algorithm for optimally $L(2, 1, 1)$-coloring a grid, with $r \geq 5$ and $c \geq 4$ or $r \geq 4$ and $c \geq 5$, is given below.

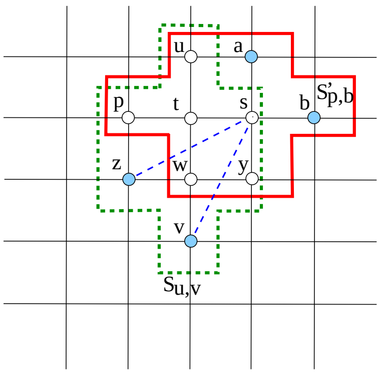

Figure 8. The subsets of vertices $S_{u,v}$ and $S'_{p,b}$ along with the square subgrid M with two dummy edges (dashed).

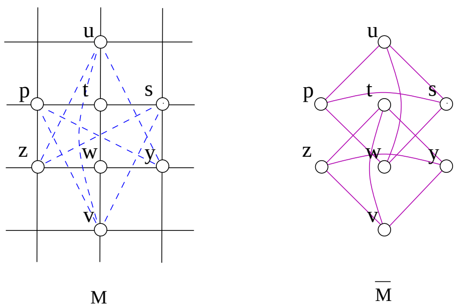

Figure 9. Subgrid M with dummy edges (dashed), and its complement $\overline{M}$.

Algorithm Grid-8-$L(2,1,1)$-coloring (G, r, c);

- If $r \geq 5$ and $c \geq 4$ or $r \geq 4$ and $c \geq 5$, then assign to each vertex $u = (i, j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } (i+j) \equiv 0 \bmod 4, i \text{ is even, and } j \text{ is even} \\ 1 & \text{if } (i+j) \equiv 0 \bmod 4, i \text{ is odd, and } j \text{ is odd} \\ 2 & \text{if } (i+j) \equiv 2 \bmod 4, i \text{ is even, and } j \text{ is even} \\ 3 & \text{if } (i+j) \equiv 2 \bmod 4, i \text{ is odd, and } j \text{ is odd} \\ 5 & \text{if } (i+j) \equiv 3 \bmod 4, i \text{ is odd, and } j \text{ is even} \\ 6 & \text{if } (i+j) \equiv 3 \bmod 4, i \text{ is even, and } j \text{ is odd} \\ 7 & \text{if } (i+j) \equiv 1 \bmod 4, i \text{ is even, and } j \text{ is odd} \\ 8 & \text{if } (i+j) \equiv 1 \bmod 4, i \text{ is odd, and } j \text{ is even} \end{cases}$$

An example of optimal coloring for a square grid of size 5×5 is illustrated in Figure 10.

Theorem 4. *The Grid-8-$L(2,1,1)$-coloring algorithm is optimal for a square grid B of size $r \times c$, with $r \geq 5$ and $c \geq 4$ or $r \geq 4$ and $c \geq 5$.*

Proof. In order to prove that the channel separation constraint is verified, one notes that

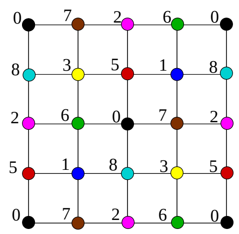

Figure 10. Optimal coloring obtained by the Grid-8-$L(2, 1, 1)$-coloring algorithm.

two consecutive colors cannot be assigned to two adjacent vertices. For example, consider the pair of colors 2 and 3. A vertex $u = (i, j)$ gets color 2 if and only if both i and j are even, and $i + j \equiv 2 \bmod 4$, while a vertex $v = (h, k)$ gets color 3 if and only if both h and k are odd, and $h + k \equiv 2 \bmod 4$. Therefore, the distance between the vertices u and v is at least 2. An analogous argument can be repeated for any pair of consecutive colors c and $c + 1$, with $0 \leq c \leq 8$.

To show that the co-channel reuse constraint holds, one notes that two vertices $u = (i, j)$ and $v = (h, k)$ are colored the same if and only if their distance $d(u, v) = 4$, and both $|i - h|$ and $|j - k|$ are even. The optimality follows from Lemma 8. □

3.1. Special Cases

Note that the square grid was defined with $r \geq 2$ and $c \geq 2$ because for $r = 1$ or $c = 1$ it reduces to a bus. For the sake of simplicity, in the following it is assumed that $r \geq c \geq 2$. Note that such an assumption is not restrictive because a square grid of size $c \times r$ can be obtained by transposition from one of size $r \times c$.

Lemma 9. *There is a k-$L(2, 1)$-coloring of a square grid B of size $r \times c$ only if*

$$k \geq \begin{cases} 4 & \textit{if } r = 2 \textit{ and } c = 2 \\ 5 & \textit{if } r \geq 3 \textit{ and } c = 2 \end{cases}$$

Proof. By Lemma 1, there is no 3-$L(2, 1)$-coloring for a square grid of size 2×2. The lower bound for k when $r \geq 3$ and $c = 2$ follows immediately because there is at least a vertex of B with degree 3 that cannot be colored either 0 or 4. Hence, from Lemma 2, $\lambda(B) \geq 5$. □

An optimal 4-$L(2, 1)$-coloring of a square grid H of size 2×2 assigns colors to vertices as follows: $f(0, 0) = 0, f(0, 1) = 4, f(1, 0) = 3, f(1, 1) = 1$. Moreover, an algorithm for optimally 5-$L(2, 1)$-coloring a square grid B whose size is $r \times 2$ with $r \geq 3$ is the following.

Algorithm Small-Grid-$L(2, 1)$-coloring (B, r, c);

- If $r \geq 3$ and $c = 2$, assign to each vertex $u = (i, j)$ the color

$$f(u) = (2i + 3j) \bmod 6$$

Theorem 5. *The Small-Grid-$L(2,1)$-coloring algorithm is optimal.*

Proof. When $c = 2$, the channel separation constraint is easily satisfied because the colors assigned to two adjacent vertices are at least 2 apart. Moreover, the co-channel reuse constraint holds because two vertices get the same color if they belong to the same column and they are at distance 3. Optimality follows from Lemma 9. $\square$

Lemma 10. *There is a k-$L(2,1,1)$-coloring of a square grid B of size $r \times c$ only if*

$$k \geq \begin{cases} 4 & \textit{if } r = 2 \textit{ and } c = 2 \\ 5 & \textit{if } r = 3 \textit{ and } c = 2 \\ 6 & \textit{if } r \geq 4 \textit{ and } c = 2 \\ 7 & \textit{if } 3 \leq r \leq 6 \textit{ and } c = 3 \\ 7 & \textit{if } r = 4 \textit{ and } c = 4 \\ 8 & \textit{if } r \geq 7 \textit{ and } c = 3 \end{cases}$$

Proof. Since any optimal $L(2,1,1)$-coloring uses at least as many colors as an optimal $L(2,1)$-coloring, it follows from Lemma 9 that $\lambda(B) \geq 4$ and $\lambda(B) \geq 5$ for square grids of size 2×2 and 3×2, respectively.

Given a square grid B of size 4×2, the two pairs of vertices $(3,0)$-$(0,1)$ and $(3,1)$-$(0,0)$ must be colored the same to satisfy the co-channel reuse distance constraint using as few colors as possible. This is equivalent to add two dummy edges, between the two pairs above, in the subgrid B_S induced by $S = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)\}$. Therefore, $\lambda(B) \geq 6$ follows from Lemma 1.

For a square grid B of size 3×3, the vertices $S = \{(0,0),(0,1),(0,2)(1,0),(1,1),(1,2),(2,1)\}$ form a clique in to the augmented graph $G_{B,4}$. Moreover, the colors assigned to the vertices $(1,0)$ and $(0,2)$ (resp., $(1,2)$ and $(0,0)$) must be at least two apart because vertices $(2,0)$ and $(0,2)$ (resp., $(2,2)$ and $(0,0)$) must be colored the same to satisfy the channel separation constraint. Hence, in the subgrid B_S induced by S four dummy edges must be added between the vertices $(1,0)$ and $(0,2)$, $(2,1)$ and $(0,2)$, $(1,2)$ and $(0,0)$, $(2,1)$ and $(0,0)$. Therefore, from Lemma 1, $\lambda(B) \geq 7$.

To prove that $\lambda(B) \geq 8$ when the size is $r \times 3$, with $r \geq 7$, the following properties are useful:

1. For any pair of vertices $u = (i,1)$ and $v = (i+3,1)$, with $0 \leq i \leq r-5$, the subset of vertices $S_{u,v} = \{u,t,w,v,p,z,s,y\}$ forms a clique in $G_{B,4}$ (see Figure 11).
2. Given the vertices t and k and the associated sets of vertices $S_{u,v}$ and $S_{t,k}$, let $U_{u,v} = \{p,u,s\} \subset S_{u,v}$ and $D_{t,k} = \{e,k,d\} \subset S_{t,k}$. The vertices in $U_{u,v}$ and $D_{t,k}$ get the same colors.

Property 1 follows from the fact that $S_{u,v}$ is a clique in the augmented graph $G_{B,4}$, while Property 2 is a consequence of the fact that $S_{u,v}$ and $S_{t,k}$ require 8 colors and that the 5 vertices t,v,w,z,y belong to both $S_{u,v}$ and $S_{t,k}$. Therefore, the remaining 3 colors must be used to color both $U_{u,v}$ and $D_{t,k}$.

Consider now the subgraph $B_{S_{t,k}}$ induced by $S_{t,k}$. By Property 2, in the subsets $S_{u,v}$ and $S_{t,k}$, the pairs of vertices t-e, t-k, and t-d must get colors which are at least 2 apart. Therefore, three dummy edges between such pairs of vertices must be added into

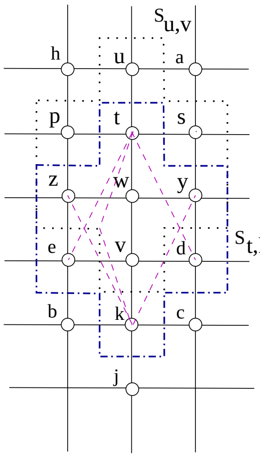

Figure 11. The subgraph $B_{S_{t,k}}$ and the dummy edges (dashed).

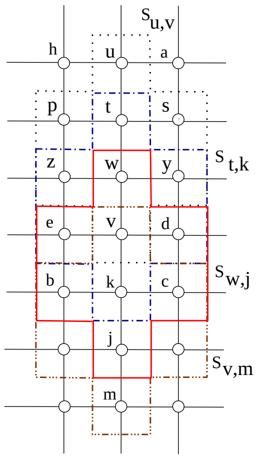

Figure 12. A case study for a 7-$L(2,1,1)$-coloring for a square grid of size 7×3.

$B_{S_{t,k}}$. Similarly, by Property 2, three dummy edges between the pairs of vertices k-z, k-t, and k-y must be added into $B_{S_{t,k}}$. The subgraph $B_{S_{t,k}}$ with these five dummy edges is depicted in Figure 11.

Now, let $\overline{B}_{S_{t,k}}$ be the complement of $B_{S_{t,k}}$ with the dummy edges. There is a Hamilton path of $\overline{B}_{S_{t,k}}$ only if the vertices z and d or y and e get two consecutive colors. Indeed, suppose by contradiction that there is a Hamilton path of $\overline{B}_{S_{t,k}}$ in which the pairs of vertices z and d or y and e are not adjacent. If there is such a Hamilton path, then there exists also a Hamilton path of the subgraph $\overline{B}_{S_{t,k}}$ including the two new dummy edges z-d, and y-e. But this is a contradiction because this latter subgraph consists of two connected components. Therefore, from Lemma 1, any 7-$L(2,1,1)$-coloring of $B_{S_{t,k}}$ uses two consecutive colors for either vertices z and d or y and e.

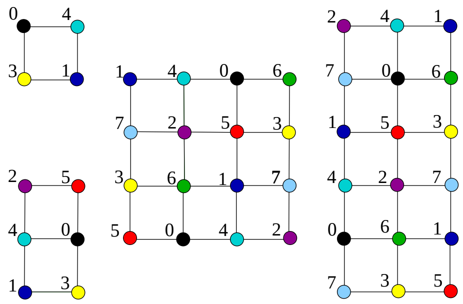

Figure 13. The optimal $L(2,1,1)$-colorings for the square grids of size 2×2, 3×2, 4×4, and 6×3.

Consider a 7-$L(2,1,1)$ coloring for $B_{S_{t,k}}$, and assume without loss of generality that $f(d) = f(z) + 1$. In order to color the grid of size 7×3, depicted in Figure 12, $U_{u,v}$ and $D_{t,k}$ must get the same colors by Property 1. Then, either $f(p) = f(d)$ or $f(p) = f(k)$ must hold to satisfy the co-channel reuse distance constraint. However, to satisfy the channel separation constraint, $f(p) = f(k)$ because p and z are adjacent. Hence, $f(s) = f(e)$ and $f(u) = f(d)$ must result. Moreover, from Property 1, $D_{w,j}$ must be colored the same as $U_{t,k}$. Now, vertex c may be colored with either $f(t)$ or $f(z)$. Again, to satisfy the channel separation constraint, $f(c) = f(t)$ follows. Thus, $f(b) = f(y)$ and $f(j) = f(z)$ must result. Finally, also the set of vertices $D_{v,m}$ must be colored as $U_{w,k}$. However, the color $f(d) = f(z) + 1$ cannot be reused in $D_{v,m}$ because all the vertices in $D_{v,m}$ are adjacent to a vertex colored $f(z)$. Hence, $\lambda(B) \geq 8$ when $r \geq 7$ and $c = 3$. □

The optimal $L(2,1,1)$-colorings for square grids of size 2×2, 3×2, 4×4, and 6×3 are depicted in Figure 13. Instead, the algorithm for optimally coloring small grids, with $r \geq 4$ and $c = 2$, or $r \geq 7$ and $c = 3$ is given below.

Algorithm Small-Grid-$L(2,1,1)$-coloring (G, r, c);

- If $r \geq 4$ and $c = 2$, assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } i \equiv 0 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 3 \bmod 6 \text{ and } j = 1 \\ 1 & \text{if } i \equiv 2 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 5 \bmod 6 \text{ and } j = 1 \\ 2 & \text{if } i \equiv 4 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 1 \bmod 6 \text{ and } j = 1 \\ 4 & \text{if } i \equiv 3 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 0 \bmod 6 \text{ and } j = 1 \\ 5 & \text{if } i \equiv 1 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 4 \bmod 6 \text{ and } j = 1 \\ 6 & \text{if } i \equiv 5 \bmod 6 \text{ and } j = 0, \text{ or } i \equiv 2 \bmod 6 \text{ and } j = 1 \end{cases}$$

- If $r \geq 7$ and $c = 3$, apply Algorithm Grid-8-$L(2,1,1)$-coloring.

Theorem 6. *The Small-Grid-$L(2,1,1)$-coloring algorithm is optimal for a square grid B of size $r \times c$, when $r \geq 4$ and $c = 2$ or $r \geq 7$ and $c = 3$.*

Proof. When $c = 2$, both the co-channel reuse and channel separation constraints are verified because the vertices of the first column are repeatedly assigned to the sequence

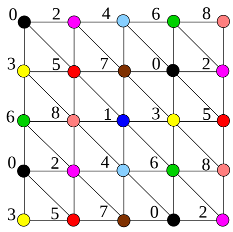

Figure 14. A cellular grid C of size 5×5 colored by the Cellular-8-$L(2,1)$-coloring algorithm.

of colors $0, 5, 1, 4, 2, 6$, while each vertex $(i, 1)$ of the second column copies the color used at the vertex $(i-3, 0)$ of the first column. When $c = 3$, the correctness follows from Theorem 4, and the optimality from Lemma 10. $\square$

In the case that the vertices of the square grid do not know their relative positions, such positions can be computed as seen in Subsection 2.2 for the honeycomb grids, with the exception that if v is a West neighbour of u and $i = 0$, then u computes its position $(0, j+1)$ and sends $CM(u, g_u, 0, j+1)$.

4. Cellular Grids

A *cellular grid* C of size $r \times c$, with $r \geq 2$ and $c \geq 2$, is obtained from a square grid B of the same size augmenting the set of edges with left-to-right *diagonal* connections. Specifically, each vertex $u = (i, j)$ of C is also connected to the vertices $v = (i-1, j-1)$ and $z = (i+1, j+1)$. Hence, each vertex has degree 6, except for the vertices on the borders.

This section reviews the optimal solutions for the $L(2,1)$- and $L(2,1,1)$-coloring problems on cellular grids published in [8,9,30].

Lemma 11. *There is a k-$L(2,1)$-coloring for a cellular grid C of size $r \times c$, with $r \geq 5$ and $c \geq 3$, or $r \geq 3$ and $c \geq 5$, or $r \geq 4$ and $c \geq 4$, only if $k \geq 8$.*

Proof. Since there are at least three vertices with degree 6 which must all get different colors, $\lambda(C) \geq 8$ follows from Lemma 2. $\square$

Algorithm Cellular-8-$L(2,1)$-coloring (C);
• If $r \geq 4$ and $c \geq 4$, or $r = 3$ and $c \geq 5$, or $r \geq 5$ and $c = 3$, assign to each vertex $u = (i, j)$ the color $$f(u) = (3i + 2j) \bmod 9$$

Figure 14 illustrates the optimal coloring obtained by means of the Cellular-8-$L(2,1)$-coloring algorithm described above.

Theorem 7. *The Cellular-8-$L(2,1)$-coloring algorithm is optimal for cellular grids of size $r \times c$, with $r \geq 4$ and $c \geq 4$, or $r = 3$ and $c \geq 5$, or $r \geq 5$ and $c = 3$.*

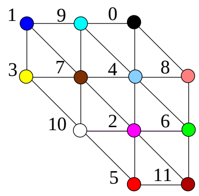

Figure 15. The subgraph D of C whose vertices form a clique in $G_{C,4}$, and an optimal 11-$L(2,1,1)$-coloring for it.

Proof. By construction, the Cellular-8-$L(2,1)$-coloring algorithm verifies the channel separation constraint. In order to prove that the co-channel reuse distance is 3, consider, without loss of generality, a vertex $u = (i,j)$ and all the vertices, on the right of u, lying on the rows $i,\ldots,i+3$ of C. Among them, the vertices which are closest to u and colored as u are $(i,j+9), (i+1,j+3), (i+2,j+7), (i+3,j)$. Thus, all of them are at distance at least 3 from u. Finally, since the algorithm uses as few colors as required by Lemma 11, the coloring is optimal. □

Lemma 12. *There is a k-$L(2,1,1)$-coloring of a cellular grid C of size $r \times c$, with $r \geq 4$ and $c \geq 4$, only if $k \geq 11$.*

Proof. Given the cellular grid $C = (V,E)$, consider the augmented graph $G_{C,4} = (V,E')$ and the subgraph D of C illustrated in Figure 15. All the 12 vertices of D are mutually at distance 3 or less, and they form a clique in $G_{C,4}$. Hence, they must be assigned to all different colors, and $\lambda(C) \geq 11$. □

Figure 15 shows how to color the subgraph D in such a way that the channel separation constraint is verified for every two adjacent vertices. Moreover, Figure 16 shows a complete coloring of a cellular grid C obtained by replicating the coloring for the subgraph D. Note that the channel separation constraint is verified not only for the vertices belonging to each copy of D, but also for the vertices belonging to the borders of two contiguous copies of D. Formally, the coloring of a cellular grid can be described as follows.

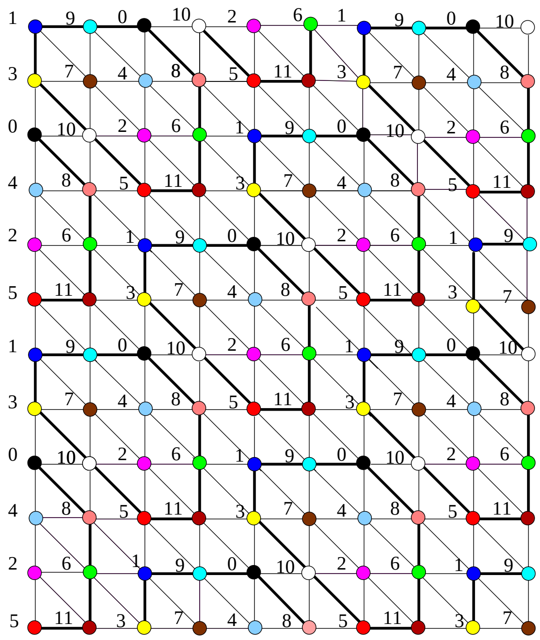

Figure 16. Optimal 11-$L(2,1,1)$-coloring for a cellular grid C.

Algorithm Cellular-11-$L(2,1,1)$-coloring (C);

- If $r \geq 4$ and $c \geq 4$, assign to each vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } (i+j) \equiv 2 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 1 & \text{if } (i+j) \equiv 0 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 2 & \text{if } (i+j) \equiv 4 \bmod 6, i \text{ is even, and } j \text{ is even} \\ 3 & \text{if } (i+j) \equiv 1 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 4 & \text{if } (i+j) \equiv 3 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 5 & \text{if } (i+j) \equiv 5 \bmod 6, i \text{ is odd, and } j \text{ is even} \\ 6 & \text{if } (i+j) \equiv 5 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 7 & \text{if } (i+j) \equiv 2 \bmod 6, i \text{ is odd, and } j \text{ is odd} \\ 8 & \text{if } (i+j) \equiv 4 \bmod 6, i \text{ is odd, and } j \text{ is odd} \\ 9 & \text{if } (i+j) \equiv 1 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 10 & \text{if } (i+j) \equiv 3 \bmod 6, i \text{ is even, and } j \text{ is odd} \\ 11 & \text{if } (i+j) \equiv 0 \bmod 6, i \text{ is odd, and } j \text{ is odd} \end{cases}$$

Theorem 8. *The Cellular-11-$L(2,1,1)$-coloring algorithm is optimal for cellular grids of size $r \times c$, with $r \geq 4$ and $c \geq 4$.*

Proof. In order to prove that the channel separation constraint is verified, it is useful to introduce the *Manhattan distance* $m(u,v)$ between any two vertices u and v, where $m(u,v)$ is the length of a shortest path between u and v including only horizontal and vertical edges, thus excluding diagonal edges. Now, any two consecutive colors are considered and it will be proved that such colors cannot be assigned to two adjacent vertices. For example, consider the pair of colors 2 and 3. A vertex $u = (i, j)$ gets the color 2 if and only if both i and j are even, and $i + j \equiv 4 \bmod 6$, while a vertex $v = (h, k)$ is colored 3 if and only if h is odd, k is even, and $h + k \equiv 1 \bmod 6$. The vertices u and v might belong to the same column, but to different rows. In this case, their distance is at least 3. In the case that they do not belong to the same column, they have Manhattan distance $m(u, v) = 3$. Hence, the vertex v which is closest to u and assigned to color 3 is $v = (i + 1, j + 2)$, as illustrated in Figure 16. Keeping track of the diagonal edges, the actual distance $d(u, v)$ is 2, and therefore the channel separation constraint is still verified. An analogous argument can be repeated for any pair of consecutive colors c and $c + 1$, with $0 \leq c \leq 10$.

To show that the co-channel reuse constraint holds, one notes that two vertices $u = (i, j)$ and $v = (h, k)$ get the same color if and only if their Manhattan distance $m(u, v) = 6$, and both $|i - h|$ and $|j - k|$ are even. Due to the diagonal edges, the actual distance $d(u, v)$ is at least 4. Indeed, the actual distance $d(u, v)$ could be 3 when $m(u, v) = 6$, but in this case $|i - h|$ and $|j - k|$ cannot be both even. The optimality follows from the lower bound shown in Lemma 12. □

4.1. Special Cases

As for square grids, in the following it is assumed that $r \geq c \geq 2$, because a cellular grid of size $c \times r$ can also be obtained by transposition from one of size $r \times c$. Note that for $c = 1$ the cellular grid reduces to a bus.

Lemma 13. *There is an optimal $L(2, 1)$- and $L(2, 1, 1)$-coloring for a cellular grid C of size $r \times c$ if and only if*

$$\lambda(C) = \begin{cases} 5 & \text{if } 2 \leq r \leq 3 \text{ and } c = 2 \\ 6 & \text{if } r \geq 4 \text{ and } c = 2 \\ 7 & \text{if } r = 3 \text{ and } c = 3 \end{cases}$$

Proof. By Lemma 1, for a cellular grid of size 2×2, $\lambda(C) \geq 5$. An optimal 5-coloring for C of size 3×2 is as follows: $f(0, 0) = 3, f(0, 1) = 1, f(1, 0) = 0, f(1, 1) = 5, f(2, 0) = 4, f(2, 1) = 2$.

In a cellular grid C of size $r \times 2$, with $r \geq 4$, there are at least 3 vertices of degree 3. Therefore, by Lemma 2, $\lambda(C) \geq 6$. An optimal 6-coloring for C is given by Algorithm Small-Cellular-coloring.

When $r = c = 3$, there is a vertex of degree 6 in C. Therefore, by Lemma 2, $\lambda(C) \geq 7$. An optimal 7-coloring is $f(0, 0) = 7, f(0, 1) = 2, f(0, 2) = 1$, $f(1, 0) = 4, f(1, 1) = 0, f(1, 2) = 5$, $f(2, 0) = 1, f(2, 1) = 6, f(2, 2) = 3$. □

Algorithm Small-Cellular-coloring (C, r, c);

- If $r \geq 4$ and $c = 2$, assign to each vertex $u = (i, j)$ the color

$$f(u) = (3i + 2j) \bmod 7$$

Lemma 14. *There is an optimal $L(2,1)$-coloring for a cellular grid C of size 4×3 if and only if $\lambda(C) = 8$.*

Proof. To derive the lower bound on $\lambda(C)$ observe that C contains two stars of degree 6, whose centers are $u = (1,1)$ and $v = (2,1)$, respectively. Since each vertex of the two stars must get a different color, vertices $(2,0)$, $(3,1)$ and $(3,2)$ must be colored as vertices $(0,0)$, $(0,1)$ and $(1,2)$. That is, the color assigned to u must be at least 2 apart from the colors assigned to $(2,0)$, $(3,1)$ and $(3,2)$, and therefore there are 3 dummy edges between u and these vertices. Thus, considering the star centered in v, $\lambda(C) \geq 8$ follows from Lemma 1. C can be colored applying Algorithm Cellular-8-$L(2,1)$-coloring. $\square$

Lemma 15. *There is an optimal $L(2,1,1)$-coloring for a cellular grid C of size $r \times 3$, with $r \geq 4$, if and only if $\lambda(C) = 9$.*

Proof. The lower bound on $\lambda(C)$ derives from the existence of a clique of size 10 in the augmented graph $G_{C,4}$. An optimal 9-$L(2,1,1)$-coloring is provided by Algorithm Small-Cellular-$L(2,1,1)$-coloring. $\square$

Algorithm Small-Cellular-$L(2,1,1)$-coloring (C, r, c);

- If $r \geq 4$ and $c = 3$, assign to vertex $u = (i,j)$ the color

$$f(u) = \begin{cases} 0 & \text{if } i \equiv 2 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 9 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 6 \bmod 10 \text{ and } j = 2 \\ 1 & \text{if } i \equiv 0 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 7 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 4 \bmod 10 \text{ and } j = 2 \\ 2 & \text{if } i \equiv 8 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 5 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 2 \bmod 10 \text{ and } j = 2 \\ 3 & \text{if } i \equiv 6 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 3 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 0 \bmod 10 \text{ and } j = 2 \\ 4 & \text{if } i \equiv 1 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 8 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 5 \bmod 10 \text{ and } j = 2 \\ 5 & \text{if } i \equiv 3 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 0 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 7 \bmod 10 \text{ and } j = 2 \\ 6 & \text{if } i \equiv 5 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 2 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 9 \bmod 10 \text{ and } j = 2 \\ 7 & \text{if } i \equiv 7 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 4 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 1 \bmod 10 \text{ and } j = 2 \\ 8 & \text{if } i \equiv 9 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 6 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 3 \bmod 10 \text{ and } j = 2 \\ 9 & \text{if } i \equiv 4 \bmod 10 \text{ and } j = 0, \text{ or } i \equiv 1 \bmod 10 \text{ and } j = 1, \\ & \text{or } i \equiv 8 \bmod 10 \text{ and } j = 2 \end{cases}$$

Finally, note that, when the vertices do not initially know their relative position within the cellular grid, a distributed algorithm can again be executed which requires optimal time and number of messages. The computation is similar to that of square grids: it still starts from vertex $(0,0)$, but it propagates along the “diagonals" of the grid.

Table 1. Minimum number $\lambda(G)+1$ of channels used for a sufficiently large network G.

Network G	$L(1)$	$L(0,1)$	$L(1,1)$	$L(2,1)$	$L(1,1,1)$	$L(2,1,1)$
Honeycomb grid	2	3	4	6	6	7
Square grid	2	4	5	7	8	9
Cellular grid	3	6	7	9	12	12
References	folklore	[4,21]	[3,5]	[8,13,18,30]	[5]	[8,30]

5. Conclusions

This paper has considered a graph theoretical approach for the Minimum-Span Fixed Channel Assignment (FCA) problem on a flat region without geographical barriers, where the wireless network stations, placed according to a plane tessellation made by regular polygons, receive a single channel per station. Precisely, after recalling some preliminary graph theoretical results, simple algorithms, based on periodic and arithmetic rules, were surveyed which optimally solve the $L(2,1)$- and $L(2,1,1)$-coloring problems for the honeycomb, square, and cellular grids, which correspond to the regular plane tessellations based on hexagons, squares, and triangles, respectively.

The results surveyed in this paper are summarized in Table 1, which indicates the minimum number of channels used for honeycomb, square, and cellular grids, not only for the $L(2,1)$- and $L(2,1,1)$-coloring problems but also for the $L(1)$-, $L(0,1)$-, $L(1,1)$- and $L(1,1,1)$-coloring problems. The channel assigned to any vertex can be computed locally provided that the relative position of the vertex in the network is known. Such a computation can be performed in constant time for all the networks.

For the sake of completeness, it is worth mentioning that the $L(\delta_1,\delta_2)$- and $L(\delta_1,1,\ldots,1)$-coloring problems have been optimally solved, for both square and cellular grids, in [30] and in [9,27], respectively. In contrast, both problems remain open for honeycomb grids, for which only the $L(1,1,\ldots,1)$-coloring problem has been optimally solved [7].

The solutions in Table 1 assume that a single channel is assigned to each station. However, by standard techniques, the proposed solutions can be readily generalized to derive sub-optimal solutions for uniform multi-channel assignment and Hybrid Channel Assignment. Indeed, when the same number m of channels has to be assigned to each vertex, the above solutions can be extended as follows. Assume that $\lambda+1$ colors are used in total and that a vertex gets the color i, then such a vertex receives also colors $i+\lambda+1, i+2(\lambda+1),\ldots,i+(m-1)(\lambda+1)$. Moreover, such a uniform multi-channel solution can be used to determine the channels in the fixed set used by a HCA strategy. Instead, additional work is needed to extend the solutions to the FCA with borrowing as well as to the DCA strategies. For instance, in DCA, the channels are often partitioned into groups, while the base stations are partitioned into clusters. Base stations can try in a distributed way to get a free channel group that is not held by one of its neighbors [11]. Usually groups have no structure other than to be a set of disjoint channels. Our approach can provide groups with guaranteed separations among the channels in the group in order to help base stations to dynamically select the channels to be used within the clusters.

It is worthy to note that the optimal solutions illustrated in this paper for the $L(2,1,1)$-coloring problem on cellular grids use as few colors as the $L(1,1,1)$-coloring problem on the same networks. Similarly, the $L(\delta_1,1,\ldots,1)$-coloring problem on square grids has been optimally solved using as few colors as the $L(1, 1, \ldots, 1)$-coloring problem on the same networks [9] when $\delta_1 \leq \lfloor \frac{\sigma-1}{2} \rfloor$. In other words, whenever no extra channels are needed to satisfy the separation constraints, using channel separation is always better than adding guard frequencies between adjacent channels. Indeed, suppose that the bandwidth of a single channel is β and that the bandwidth of a guard frequency is γ. Consider a channel assignment problem with co-channel reuse distance σ. If the $L(1, 1, \ldots, 1)$-coloring problem is optimally solved, say using $\lambda + 1$ colors, and then a guard frequency is added between adjacent channels to handle the adjacent frequency interference problem, then the overall bandwidth used is

$$W_{\text{guard}} = (\lambda + 1)\beta + \lambda\gamma.$$

In contrast, if channel separation is introduced as required by the $L(\delta_1, 1 \ldots, 1)$-coloring problem, say using $\lambda' + 1$ colors, then the total bandwidth used is

$$W_{\text{separation}} = (\lambda' + 1)\beta.$$

Clearly, if $\lambda = \lambda'$, then $W_{\text{separation}} < W_{\text{guard}}$, which implies that using channel separation is better than using guard frequency. As mentioned above, this happens for the $L(\delta_1, 1, \ldots, 1)$-coloring problem on square grids, as well as for the $L(2, 1, 1)$-coloring problem on cellular grids. If $\lambda < \lambda'$, the channel separation technique may or may not be more appealing than the guard frequency technique, depending on the values of γ. For example, consider the $L(1, 1, 1)$- and the $L(2, 1, 1)$-coloring problems on a square grid. By the above reasoning one obtains

$$W_{\text{guard}} = 8\beta + 7\gamma \;\text{ and }\; W_{\text{separation}} = 9\beta,$$

which implies that using channel separation is better than adding guard frequency when $\gamma \geq \frac{1}{7}\beta$.

From a theoretical point of view, it remains as an interesting open question to solve the general $L(\delta_1, \delta_2, \ldots, \delta_{\sigma-1})$-coloring problem, with $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{\sigma-1}$, on honeycomb, square, or cellular grids.

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