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Channel assignment for interference avoidance in honeycomb wireless networks ☆

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Abstract

The large development of wireless services and the scarcity of the usable frequencies require an efficient use of the radio spectrum which guarantees interference avoidance. The channel assignment problem (CAP) achieves this goal by partitioning the radio spectrum into disjoint channels, and assigning channels to the network base stations so as to avoid interference. On a flat region without geographical barriers and with uniform traffic load, the network base stations are usually placed according to a regular plane tessellation, while the channels are permanently assigned to the base stations. This paper considers the CAP problem on the honeycomb grid network topology, where the plane is tessellated by regular hexagons. Interference between two base stations at a given distance is avoided by forcing the channels assigned to such stations to be separated by a gap which is proportional to the distance between the stations. Under these assumptions, the CAP problem on honeycomb grids can be modeled as a suitable coloring problem. Formally, given a honeycomb grid $G=(V,E)$ and a vector $(\delta_1, \delta_2, \ldots, \delta_t)$ of positive integers, an $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of G is a function f from the vertex set V to a set of nonnegative integers such that $|f(u)-f(v)| \geqslant \delta_i$, if $d(u,v)=i$, $1 \leqslant i \leqslant t$, where $d(u,v)$ is the distance (i.e. the minimum number of edges) between the vertices u and v. An optimal $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring for G is one minimizing the largest used integer over all such colorings. This paper presents efficient algorithms for finding optimal $L(2,1)$-, $L(2,1^2)$- and $L(1^t)$-colorings of honeycomb grids. Such colorings use less colors than those needed by the cellular and square grids with the same number of vertices.

Keywords: Wireless networks; Interference; Channel assignment; Separation; Honeycomb grids; $L(2,1)$-coloring; $L(2,1^2)$-coloring; $L(1^t)$-coloring

1. Introduction

In a wireless network, the main difficulty against an efficient use of the radio spectrum is given by interference, caused by unconstrained simultaneous transmissions, which result in damaged communications. The channel assignment problem (CAP) models the job of efficiently assigning the radio spectrum to the set of base stations of the network. Such a problem, that first appeared in TV broadcasting and military communications in late 1960s, keeps renewing its interest due to the large development of wireless telephone networks (e.g. FDMA, TDMA, GSM networks) and satellite communication [1]. Although there are many different models, all scenarios are characterized by a set of transmitters (usually, antennae), a set of disjoint channels (frequencies) obtained partitioning the radio spectrum, and a strategy for assigning channels to transmitters so that data communications are possible.

The channel assignment can be done following several strategies [14]. In the fixed channel assignment

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(FCA), channels are statically assigned to the transmitters for their exclusive and permanent use, and remain stable over time [1,5,17]. Opposite to FCA, dynamic channel assignment (DCA) maintains all channels in a central pool, and dynamically assigns them to the transmitters for temporary use [7,10,11]. Finally, hybrid channel assignment (HCA) combines the two above strategies [14]. FCA performs well when the traffic load is uniform in time and in space, since it yields maximum channel reusability. In contrast, DCA is more suited in the case of short-term temporal and spatial traffic variations, since it privileges the flexibility of the channel allocation with respect to the channel reusability. Under mixed traffic conditions, either HCA or FCA with borrowing are used. HCA, which has channels partitioned into fixed and dynamic sets, performs well when on the top of a constant traffic load there is a fraction of highly variable communications. In the FCA with borrowing, a transmitter which has used all its statically assigned channels can occasionally borrow free channels from its neighboring transmitters.

This paper concentrates on FCA. Using this technique, the CAP can be modeled as variants of vertex graph coloring (for surveys, see [1,13,18]). Formally, an undirected graph $G = (V, E)$ models the wireless network, where the vertices in V represent the transmitters and the edges in E represent pairs of transmitters that may potentially interfere. The separation required to avoid interference between the frequencies assigned to the edge end-points is represented by a label of the edge. Colors (i.e. frequencies) have to be assigned to the vertices so that the separation constraints are verified and an objective function is optimized. Typical objective functions range from minimizing the difference between the largest and the lowest used colors while avoiding interference (called, minimum-span) to minimizing interferences using a given number of colors (called, fixed-spectrum) [1,16]. Moreover, typical values for the separation labels are upper bounded by 3 [1].

For arbitrary network topologies and general separation constraints, the resulting vertex coloring problems are computationally intractable (i.e., NP-hard). Therefore, the CAP problem is usually addressed by means of heuristic approaches, like genetic algorithms, taboo search, saturation degree, simulated annealing, and ants heuristics, just to name a few [1]. The performance of such heuristics is compared on widely accepted benchmarks, like CELAR data, COST 259 data, and Philadelphia instances. In particular, the Philadelphia instances, that have been heuristically solved to optimal for the minimum-span objective function, suggest the relevance of topologies based on regular tessellations of the plane. In such a case, the interference phenomena depend on the distance among the antennae. Thus, the separation constraints are modeled by a separation vector $(\delta_1, \delta_2, \ldots, \delta_t)$ of positive integers such that channels assigned to base stations at distance i be at least δ_i apart [1,13,14], which implies that the same color can be reused only at stations whose distance is larger than t. Typical values of t studied so far are upper bounded by 4, while typical values of the separations are $\delta_1 = 3$ or 2, $\delta_2 = 2$ or 1, and $\delta_3 = \delta_4 = 1$ [1]. However, in the next generation of wireless access systems, due to the decreasing cost of infrastructures and to the need of wider bandwidth, a large number of small cells, each with significant power, is expected to cover a huge communication region [25]. Therefore, the upper bound of t is expected to become much larger than 4.

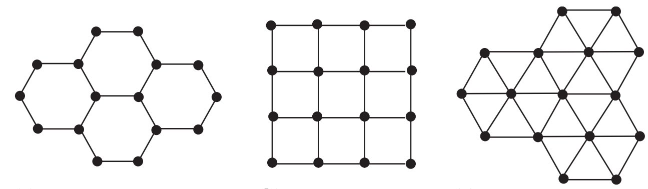

Fig. 1. The possible grids of 16 vertices: (a) honeycomb grid, (b) square grid, and (c) cellular grid.

It is worth to note that, exploiting the regularity of plane tessellation, several algorithms have been devised to find optimal solutions in polynomial time in the simplest case when a single color per vertex has to be assigned [5,20,21]. Such solutions can then be used to derive sub-optimal solutions in the more general uniform multi-coloring case, where the same number m of channels has to be assigned to each vertex. Indeed, this can be accomplished by optimally assigning one color per vertex, e.g. using s colors in total, and then coloring each vertex by m colors repeatedly shifted s channels up. Precisely, if a vertex gets the single color i, then it receives also colors $i + s, i + 2s, \ldots, i + (m - 1)s$.

It is well-known that only three different regular tessellations of the plane exist, depending on the kind of regular polygons used. Specifically, the honeycomb, square and cellular tessellations cover the plane, respectively, by regular hexagons, squares, and triangles. Such tessellations can be used to place at the polygon vertices the base stations of the wireless communication networks, leading to three well-known topologies: *honeycomb*, *square* and *cellular* grids, depicted in Fig. 1 for 16 vertices. So far, the most studied topology for wireless communication networks has been the cellular grid. However, the performance of a topology can be evaluated with respect to several parameters, such as *degree* and *diameter*. Comparing the above three grids in terms of degree and diameter, measured with respect to the same number of vertices, one notes that a honeycomb grid has the smallest degree, a cellular grid has the smallest diameter, while a square grid is always worst. However, as proved in [22], defined the network *cost* as the product of the degree and diameter, the honeycomb grid beats both the cellular and square grids, as summarized in Table 1 for grids with n vertices (in such a table, coefficients are rounded and additive constants are neglected).

This paper investigates for the first time the minimum-span FCA problem on honeycomb grid network topologies, where a single channel has to be assigned to each station. By

Table 1
Comparison of networks, each with n vertices (data are approximated) [22]

Network	Degree	Diameter	Cost
Honeycomb grid	3	$1.63\sqrt{n}$	$4.9\sqrt{n}$
Square grid	4	$2\sqrt{n}$	$8\sqrt{n}$
Cellular grid	6	$1.16\sqrt{n}$	$6.93\sqrt{n}$

the above-mentioned standard techniques, the proposed solutions can be readily generalized to uniform multi-coloring and HCA.

Formally, the channel assignment problem to be studied in this paper can be modeled as follows. Let $G = (V, E)$ be an undirected graph representing the honeycomb grid and let $(\delta_1, \delta_2, \ldots, \delta_t)$ be the separation vector. An $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring of G is a function f from the vertex set V to the set of nonnegative integers $\{0, \ldots, \lambda\}$ such that $|f(u) - f(v)| \geqslant \delta_i$, if $d(u, v) = i$, $1 \leqslant i \leqslant t$, where $d(u, v)$ is the distance (i.e. the minimum number of edges) between the vertices u and v. An *optimal* $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring for G is one minimizing λ over all such colorings. Note that, since the set of colors includes 0, the overall number of colors involved by an optimal coloring f is in fact $\lambda + 1$ (although, due to the channel separation constraint, some colors in $\{1, \ldots, \lambda - 1\}$ might not be actually assigned to any vertex). Thus, the channel assignment problem consists of finding an optimal $L(\delta_1, \delta_2, \ldots, \delta_t)$-coloring for G. Note that an $L(1)$-coloring is just a classical vertex coloring. In this case the channel assignment problem is called chromatic number problem. In case of separation vectors containing repeated integer values, a more compact notation will be convenient and so, as an example, $(\delta_1, 1^q)$ will be a shorthand for $(\delta_1, \underbrace{1, 1, \ldots, 1}_{q})$.

This paper provides efficient algorithms to find optimal solutions for the $L(2, 1)$-, $L(2, 1^2)$-, and $L(1^t)$-coloring problems on honeycomb grids. This paper follows the same graph theoretical approach first outlined in [3,4,9,15,20], which has led to several optimal solutions for particular topologies and/or separations. Although the $L(1^t)$-coloring problem, for any positive integer t, has been proved to be intractable in [15] for arbitrary graphs, optimal colorings have been proposed for rings, square grids and chordal graphs [4,2]. Moreover, optimal $L(\delta_1, 1^{t-1})$-colorings have been devised for rings, square grids and cellular grids [5,21]. In addition, optimal $L(\delta_1, \delta_2)$-colorings on square grids and cellular grids have been given in [24]. Finally, the $L(2, 1^2)$-coloring problem has been also optimally solved for square grids, cellular grids and rings [5,24], while optimal $L(2, 1)$-colorings were given in [6,8,12,19].

The rest of the present paper is structured as follows. Section 2 briefly recalls some preliminary graph theoretical results (e.g., augmented graph, clique, t-independent set) that will be used in the following sections. Moreover, a simple distributed scheme is given to allow the vertices to compute their own relative positions in the grid, in case these information are not already available. Such positions will then be used by the vertices to self-assign their proper channel in constant time. Section 3 provides simple, periodic, and arithmetic rules to optimally solve in $O(1)$ time the $L(2, 1)$- and $L(2, 1^2)$-coloring problems.

Section 4, the chest of this paper, considers the $L(1^t)$-coloring problem for arbitrary t. When t is odd, Section 4.1 first shows a lower bound on the minimum number of colors which is based on the size of a clique, called diamond. Then, once a diamond is optimally colored, an optimal grid coloring is obtained by tessellating the entire grid by replicas of the colored diamond. Two different tessellations may arise depending on the symmetry of the diamond, but in both cases each grid vertex can color itself in constant time. When t is even, the lower bound on the size of a clique is too weak. Therefore, Section 4.2 shows a stronger lower bound which is based on the size of a t-independent set. Then, it is shown how each vertex can color itself in $O(1)$ time by a simple, periodic, and arithmetic rule. Finally, conclusions are offered in Section 5, where it is also pointed out that all the solutions proposed in this paper for the honeycomb grid use less colors than those needed by the cellular and square grids with the same number of vertices.

2. Preliminaries

The $L(1)$-coloring problem on a graph G is exactly the classical vertex coloring problem on G, where the minimum number of colors needed is $\lambda + 1 = \chi(G)$, the *chromatic number* of G. Similarly, given any separation vector $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_t)$, the minimum number of colors needed to $L(\bar{\delta})$-color G will be denoted by $\chi_{\bar{\delta}}(G)$ and called the $\bar{\delta}$*-chromatic number* of G. In the special case of $L(1^t)$-colorings, the term *t-chromatic number* of G, denoted by $\chi_t(G)$, will be used. A very simple lower bound for $\chi_t(G)$ is obtained by considering the *maximum clique* (i.e. the largest complete subgraph) K of the *augmented* graph $A_{G,t}$ built up as follows. The vertex set of $A_{G,t}$ is the same as the vertex set of G, while an edge $[r, s]$ belongs to the edge set of $A_{G,t}$ iff the distance $d(r, s)$ between the vertices r and s in G satisfies $d(r, s) \leqslant t$. The role of K is apparent for deriving lower bounds on the minimum number of channels for the $L(1^t)$-coloring problem on G. Indeed, since for each pair of vertices in K there is an edge in $A_{G,t}$, all the vertices in K are mutually at distance smaller than or equal to t in G, and therefore they must be colored differently. Hence, the size of the largest clique of $A_{G,t}$, known as the *clique number* $\omega(A_{G,t})$, is a lower bound on the t-chromatic number. That is, $\chi_t(G) \geqslant \omega(A_{G,t})$.

A *t-independent set* is a subset S_t of vertices of G whose pairwise distance is at least $t + 1$. If the size of S_t is the largest possible, then S_t is a *maximum t-independent set*, and is denoted by S_t^*. Clearly, given a t-independent set S_t, all its vertices can get the same color in an $L(1^t)$-coloring. In this way, assigning different colors to different t-independent sets one obtains a feasible $L(1^t)$-coloring. Conversely, given a feasible $L(1^t)$-coloring, all the vertices with the same color form a t-independent set. A minimum $L(1^t)$-coloring uses as many colors as the minimum number of t-independent sets that cover all the vertices. Moreover, any feasible $L(1^t)$-coloring uses at least as many colors as the minimum number $\mu_t(G)$ of maximum t-independent sets that cover all the vertices, that is $\chi_t(G) \geqslant \mu_t(G)$.

Clearly, the $L(2,1)$- and $L(2,1^2)$-colorings require at least as many colors as the $L(1^2)$- and $L(1^3)$-colorings, respectively. However, in some cases, such trivial lower bounds can be improved by the following results. Let the *complement graph* $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ be the graph having the same vertex set V as G and having the edge set $\overline{E}$ obtained by swapping edges and non-edges in E. Recall that a *Hamilton path* is a path that traverses each vertex of a graph exactly once.

Lemma 1 (Griggs and Yeh [12]). *Let G be a graph of diameter t, that is having $d(u, v) \leqslant t$ for every pair of vertices u and v of G. Then, $\chi_{(2,1^{t-1})}(G) = |V(G)|$ if and only if $\overline{G}$ has a Hamilton path.*

Consider the *star* graph S_ρ which consists of a *center* vertex c with degree ρ, and ρ *ray* vertices of degree 1.

Lemma 2 (Griggs and Yeh [12]). *Let the center c of S_ρ be already colored. Then*

$$\chi_{(2,1)}(S_\rho) \geqslant \begin{cases} \rho+2 & if\ f(c)=0\ or\ f(c)=\rho+1, \\ \rho+3 & if\ 0<f(c)<\rho+1. \end{cases}$$

Let G_1 and G_2 be any two graphs, and let $V(G)$ denote the vertex set of a graph G. A *t-homomorphism* from G_1 to G_2 is a total function $\phi : V(G_1) \mapsto V(G_2)$ such that
1. $\phi(u) = \phi(v)$ only if $u = v$ or $d(u, v) > t$;
2. $d(\phi(u), \phi(v)) = d(u, v)$ if $d(u, v) < t$.

Now, if g is an $L(\delta_1, \ldots, \delta_t)$-coloring of G_2, and ϕ is a t-homomorphism from G_1 to G_2, then the composition $g \circ \phi$ is an $L(\delta_1, \ldots, \delta_t)$-coloring of G_1.

In this paper, the honeycomb grid is considered, and optimal channel assignment algorithms will be shown for sufficiently large grids. For the sake of simplicity, *brick* representations of honeycomb grids will be adopted from now on. Specifically, as shown in Fig. 2, each hexagon will be represented by a rectangle spanning 3 rows and 2 columns. In this way, a honeycomb grid H of size $n = rc$ is represented by r rows and c columns, indexed respectively, from 0 to $r-1$ (from top to bottom) and from 0 to $c-1$ (from left to right), with $r \geqslant 3$ and $c \geqslant 2$. A generic vertex u of H is denoted by $u = (i, j)$, where i is its row index and j is its column index. Note that each vertex has degree 3, except for some vertices on the borders. In particular, each vertex (i, j), which does not belong to the grid borders, is adjacent

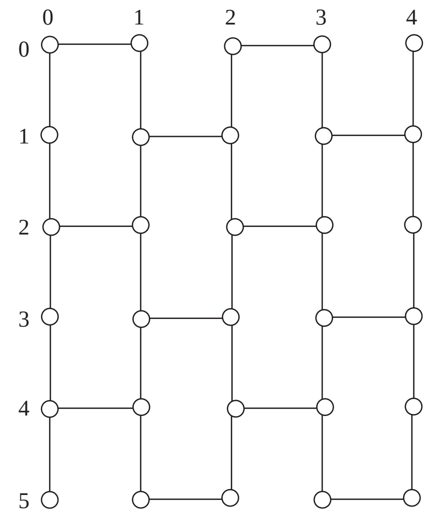

Fig. 2. A brick representation of a honeycomb grid of size 6×5.

to the following 3 vertices:

1. $\begin{cases} (i, j+1) & \text{if } i+j \text{ is even,} \\ \text{or} & \\ (i, j-1) & \text{if } i+j \text{ is odd,} \end{cases}$
2. $(i-1, j)$,
3. $(i+1, j)$.

The channel assignment algorithms to be presented allow any vertex to self-assign its proper channel in constant time provided that it knows its relative position within the honeycomb grid. If this is not the case, such relative positions can be computed for all the vertices using a simple distributed algorithm requiring optimal time and optimal number of messages, as detailed in the next subsection.

2.1. Distributed computation of relative positions

Assume that each vertex of the grid only knows its own geographic position (e.g. by means of its I.D. or a local geographic position system (GPS)) and the names of its neighbors (which can be easily obtained by the usual topology-exchange distributed algorithm [23]).

The vertices are assumed to be asynchronous and can communicate by exchanging control messages (e.g. via dedicated system signals). There is only one kind of control message, which is sent by a vertex to tell its geographic position and its relative position to all its neighbors. When a vertex receives a control message from a neighbor, it is capable of recognizing whether the sender is a North, South, East, or West neighbor, by comparing its geographic position and that of the sender (the agreement about the actual cardinality points can be established and broadcast by the vertex starting the computation, after knowing the GPS positions of its neighbors). When a vertex receives a control message from a neighbor, if it has not yet computed its position and some conditions are met, then it computes its own relative position and in turn sends a control message, otherwise it neglects the message.

The computation is started by the upper-left corner vertex in the honeycomb grid, which is the only vertex knowing its position (0,0). A control message is structured as $CM(v, g_v, i, j)$, where g_v and (i, j) are the geographic and relative positions of v, respectively. When a vertex u receives $CM(v, g_v, i, j)$ from a North neighbor v and $i \geqslant 1$, then u computes its relative position $(i + 1, j)$ and sends $CM(u, g_u, i + 1, j)$ so as to propagate the computation downwards along the columns of the honeycomb grid. In the first two rows, however, different conditions have to be dealt with. Specifically, if v is a West neighbor of u and $i = 0$, then u computes its position $(0, j + 1)$ and sends $CM(u, g_u, 0, j+1)$, while if v is a South neighbor of u and $i = 1$, then u computes $(0, j)$ and sends $CM(u, g_u, 0, j)$. As for the vertices in the second row, if v is a North neighbor and $i = 0$, then u computes $(1, j)$ and sends $CM(u, g_u, 1, j)$, while if v is a West neighbor and $i = 1$, then u computes $(1, j + 1)$ and sends $CM(u, g_u, 1, j + 1)$.

It is easy to see that the overall number of messages required is $O(rc)$ while the total time is $O(r + c)$, assuming that a message reaches its destination in $O(1)$ time. Since there are rc vertices in the grid and the grid diameter is $O(r + c)$, the channel assignment for all the vertices can be performed in a distributed fashion so as to require an optimal time and an optimal number of messages.

3. Optimal $L(2, 1)$- and $L(2, 1^2)$-colorings

This section considers the $L(2, 1)$- and $L(2, 1^2)$-coloring problems on honeycomb grids. First, lower bounds on the minimum number of colors are exhibited for both problems. Then, simple, periodic, and arithmetic rules to optimally color in $O(1)$ time each grid vertex are given.

The following lower bound holds for the $L(2, 1)$-coloring problem.

Lemma 3. *Let H be a honeycomb grid of size $r \times c$, with $r \geqslant 3$ and $c \geqslant 3$. Then $\chi_{(2,1)}(H) \geqslant 6$.*

Proof. It follows immediately from Lemma 2 since there is at least one vertex of degree 3 that cannot be colored either 0 or 4. Hence, $\chi_{(2,1)}(H) \geqslant 6$. □

Next, a simple, periodic, and arithmetic rule is proposed to $L(2, 1)$-color with 6 colors any honeycomb grid H. Given any vertex $u = (i, j)$ of H, the algorithm simply assigns in $O(1)$ time the color

$$f(u) = (2i + 3j) \bmod 6$$

to vertex u. As an example, an optimal coloring for the honeycomb grid of Fig. 2 is illustrated in Fig. 3.

Theorem 1. *The above coloring algorithm delivers a feasible $L(2, 1)$-coloring with 6 colors. This $L(2, 1)$-coloring is optimal for any honeycomb grid H of size $r \times c$, with $r \geqslant 3$ and $c \geqslant 3$.*

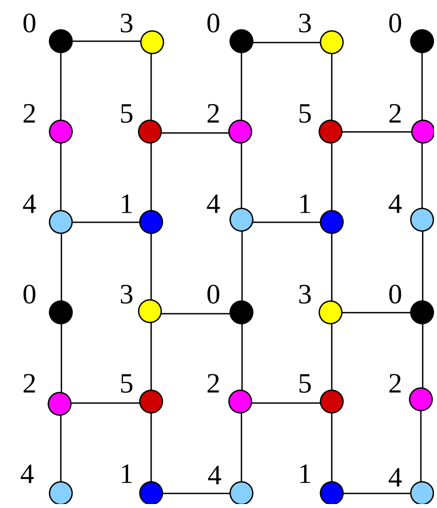

Fig. 3. Optimal $L(2, 1)$-coloring of a honeycomb grid.

Proof. Consider a generic vertex $u = (i, j)$ of H. Let $v = (h, k)$ be any vertex adjacent to u. Then, either $k = j$ and $h = i \pm 1$, whence $f(v) \equiv_6 f(u) \pm 2$, or $h = i$ and $k = j \pm 1$, whence $f(v) \equiv_6 f(u) \pm 3$, where $\equiv_m$ denotes a congruence modulo m. Consider therefore a node $v = (h, k)$ with $d(u, v) = 2$. Note that only two cases are possible:

Case 1: $k = j$ and $h = i \pm 2$. In this case, $f(v) \equiv_6 f(u) \pm 4$.

Case 2: $k = j \pm 1$ and $h = i \pm 1$. In this case, either $f(v) \equiv_6 f(u) \pm 5$ or $f(v) \equiv_6 f(u) \pm 1$.

The optimality follows from Lemma 3. □

Consider now the $L(2, 1^2)$-coloring problem. The following lower bound holds.

Lemma 4. *Let H be a honeycomb grid of size $r \times c$, with $r \geqslant 4$ and $c \geqslant 3$, or $r \geqslant 5$ and $c = 2$. Then $\chi_{(2,1^2)}(H) \geqslant 7$.*

Proof. Consider the augmented graph $A_{H,3} = (V, E')$ and the subset of vertices $S = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$. Since all the 6 vertices in S are mutually at distance at most 3 in H, they form a clique in $A_{H,3}$. Therefore, $\chi_{(2,1^2)}(H) \geqslant 6$.

In the case where $r \geqslant 4$ and $c \geqslant 3$, consider the subgraph H_S induced by S and also the vertex (3,0). To use exactly 6 colors, vertex (3,0) must get the same color as vertex (0,1). Moreover, the colors assigned to vertices (2,0) and (3,0) must have a gap of at least $\delta_1 = 2$. Hence, also the colors of (2,0) and (0,1) must have a gap of at least $\delta_1 = 2$. This is equivalent to add in H_S a *dummy* edge between vertices (2,0) and (0,1). The same reasoning can be repeated for the pairs

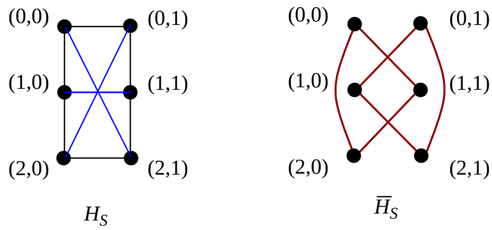

Fig. 4. The subgraph H_S with the dummy edges (dashed), and its complement $\overline{H_S}$.

of vertices (1,1) and (1,0), and (2,1) and (0,0), as illustrated in Fig. 4. Note that H_S with the dummy edges is isomorphic to a $K_{3,3}$ graph, that is a complete bipartite graph with 3 vertices per part. Now, consider the complement $\overline{H_S}$ of H_S, depicted also in Fig. 4. There is no Hamilton path in $\overline{H_S}$ since it consists of two connected components. It follows from Lemma 1 that $\chi_{(2,1^2)}(H) \geqslant 7$.

By a similar argument, the same lower bound can be proved for honeycomb grids when $r \geqslant 5$ and $c = 2$. Indeed, when $r \geqslant 5$ and $c = 2$, vertices (2,0) and (2,1) belong to two distinct cliques in $A_{H,3}$. Then, to keep $\chi_{(2,1^2)}(H) = 6$, the same colors used for vertices (0,0), (0,1), (1,0), (1,1) must be reused for vertices (3,0), (3,1), (4,0), (4,1). In particular, as explained before, vertices (3,0) and (3,1) must get the same colors as (0,1) and (0,0). Thus, vertices (4,1) and (4,0) must get the same colors as (1,0) and (1,1). As said, this is equivalent to add in H_S the three *dummy* edges between the pairs of vertices (2,0) and (0,1), (2,1) and (0,0), and (1,0) and (1,1). As before, there is no Hamilton path in $\overline{H_S}$. □

Let $N_n = \{0, \ldots, n-1\}$ denote the set of the first n natural numbers. A simple, periodic, and arithmetic rule is proposed to $L(2, 1, 2)$-color with 7 colors any honeycomb grid H.

A kernel fact in this construction is that, when considering an $L(2, 1^2)$-coloring, the graph H_S with dummy edges introduced in the proof of Lemma 4 is isomorphic to the $K_{3,3}$ graph. Moreover, recalling that $\chi_{(2,1,2)}(G) \geqslant \chi_{(2,1^2)}(G) \geqslant \chi_{(2,1)}(G)$ for any G, and observing that $\chi_{(2,1)}(K_{3,3}) \geqslant 7$ as shown in Fig. 5, it holds $\chi_{(2,1,2)}(H) \geqslant 7$.

The algorithm to $L(2, 1, 2)$-color the honeycomb grid H works in two steps.

1. Fix any feasible $L(2, 1)$-coloring $g : N_6 \mapsto N_6$ of $K_{3,3}$, for example 0,6,1,4,2,5 as in Fig. 5,
2. Assign to each vertex $u = (i, j)$ the color $f(u) = g((i + 3j) \bmod 6)$.

Clearly such an algorithm requires $O(1)$ time to color vertex u. As an example, an optimal $L(2, 1, 2)$-coloring for the honeycomb grid of Fig. 2 is illustrated in Fig. 6.

Theorem 2. *The above coloring algorithm delivers a feasible $L(2, 1, 2)$-coloring with 7 colors. This $L(2, 1, 2)$-coloring is optimal for both the $L(2, 1^2)$- and $L(2, 1, 2)$-coloring problems for any honeycomb grid H of size $r \times c$, with $r \geqslant 4$ and $c \geqslant 3$ or with $r \geqslant 5$ and $c = 2$.*

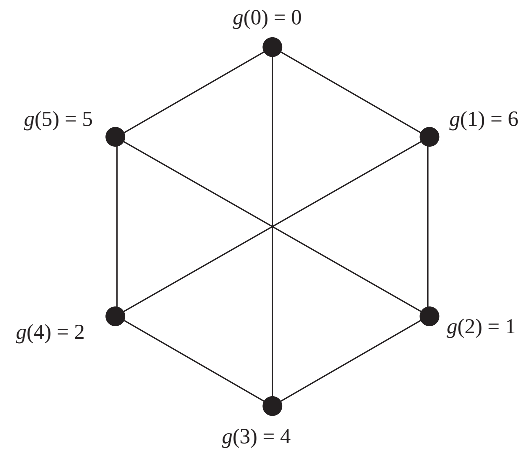

Fig. 5. An $L(2, 1)$-coloring g of $K_{3,3}$.

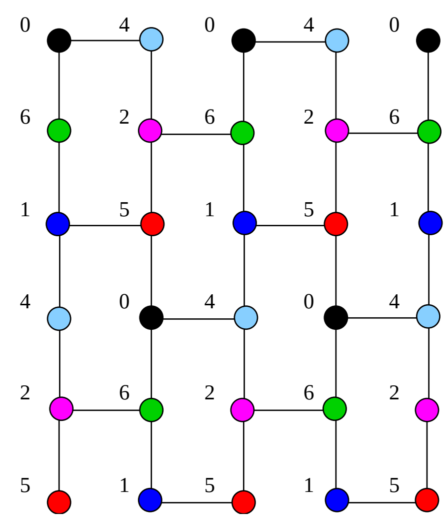

Fig. 6. Optimal $L(2, 1, 2)$-coloring of a honeycomb grid.

Proof. Let $\phi : V(H) \mapsto N_6$ be the function defined as $\phi(i, j) = (i + 3j) \bmod 6$. Such a function is a 3-homomorphism from any honeycomb grid H to $K_{3,3}$. Therefore, $f(i, j) = g((i + 3j) \bmod 6)$ is an optimal coloring by Lemma 4. □

4. Optimal $L(1^t)$-coloring

In this section, optimal solutions for the $L(1^t)$-coloring problem of sufficiently large honeycomb grids will be presented. The coloring depends on the value of $t \bmod 8$. In particular, when t is odd (i.e., $t \equiv 1, 3, 5, 7 \bmod 8$) a lower bound on the number of colors is given by the clique size of the augmented graph $A_{H,t}$ and this bound is achievable. When t is even (i.e., $t \equiv 0, 2, 4, 6 \bmod 8$), the clique size lower bound is no more achievable, and a stronger lower bound is needed, which depends on the minimum number of maximum t-independent sets.

4.1. $L(1^t)$-coloring with t odd

In this subsection, the coloring of the honeycomb grid is studied for t odd. First, a lower bound on the minimum number of colors is proved which is based on a counting argument on the size of a set of grid vertices, called diamond, at reciprocal distance no larger than t. Then, once a diamond is optimally colored, it is shown how an optimal grid coloring can be obtained by tessellating the entire grid by replicas of the colored diamond. Depending on the symmetry of the diamond, two different tessellations arise. The first tessellation, called symmetric, deals with the values $t \equiv 1, 5 \bmod 8$, while the other one, called asymmetric, involves the values $t \equiv 3, 7 \bmod 8$. For the symmetric (resp., asymmetric) case, Section 4.1.1 (resp., Section 4.1.2) shows how each grid vertex can color itself in constant time.

The following lower bound on the number of colors holds.

Lemma 5. *Let $t = 8p + q$, with $p \geqslant 0$ and $q = 1, 3, 5, 7$. There is an $L(1^t)$-coloring of a honeycomb grid H of size $r \times c$, with $r \geqslant t + 1$ and $c \geqslant \lceil \frac{t-3}{4} \rceil + \lfloor \frac{t+1}{4} \rfloor + 1$, only if*

$$\chi_t(H) \geqslant \begin{cases} 24p^2 + 12p + 2 & if\ q = 1, \\ 24p^2 + 24p + 6 & if\ q = 3, \\ 24p^2 + 36p + 14 & if\ q = 5, \\ 24p^2 + 48p + 24 & if\ q = 7. \end{cases}$$

Proof. The maximum clique of $A_{H,t}$ is a *diamond* with $\lceil \frac{t-3}{4} \rceil + \lfloor \frac{t+1}{4} \rfloor + 1$ columns. The leftmost column has $(t+1) - 2\lceil \frac{t-3}{4} \rceil$ vertices, and each subsequent column has two extra vertices up to the central column which counts $t+1$ vertices. Each of the remaining $\lfloor \frac{t+1}{4} \rfloor$ columns, on the right of the central one, decreases its size by two. In particular, the rightmost column has $(t+1) - 2\lfloor \frac{t+1}{4} \rfloor$ vertices. Depending on the value of q, the number of left columns is

$$\left\lceil \frac{t-3}{4} \right\rceil = \begin{cases} 2p & \text{if } q = 1, 3, \\ 2p+1 & \text{if } q = 5, 7, \end{cases}$$

while the number of right columns is

$$\left\lfloor \frac{t+1}{4} \right\rfloor = \begin{cases} 2p & \text{if } q = 1, \\ 2p+1 & \text{if } q = 3, 5, \\ 2p+2 & \text{if } q = 7. \end{cases}$$

Note that the shape of the maximum clique varies with q. For instance, Fig. 7 shows the maximum cliques when $q = 1, 3, 5$ and 7 and $t = 17, 19, 21$ and 23, respectively.

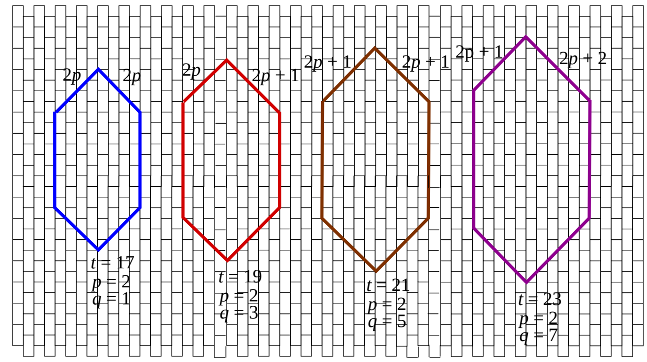

Fig. 7. The maximum cliques (diamonds) for $t = 17, 19, 21$ and 23.

In general, the size of the maximum clique is given by

$$\omega(A_{H,t}) = (t+1) + \sum_{i=1}^{\lfloor \frac{t+1}{4} \rfloor} (t+1-2i) + \sum_{i=1}^{\lceil \frac{t-3}{4} \rceil} (t+1-2i).$$

Solving the above formula with $t = 8p + q$, the proof follows. □

By the above lemma, all the vertices of each diamond must get a different color. An optimal solution for the $L(1^t)$-coloring problem, with t odd, can be easily achieved tessellating the honeycomb grid by means of diamonds, all colored in the same way. Observing Fig. 7, one notes that diamonds have the same number of left and right columns, i.e. they are symmetric, for $q = 1, 5$; while they have one more right column, i.e., they are asymmetric, for $q = 3, 7$. Therefore, there are two possible tessellations depending on the symmetry of the diamonds, which are illustrated in Fig. 8 (where $t = 13$ and 15 are assumed).

Theorem 3. *When t is odd, the above algorithms lead to optimal $L(1^t)$-colorings for a honeycomb grid H of size $r \times c$, with $r \geqslant t + 1$ and $c \geqslant \lceil \frac{t-3}{4} \rceil + \lfloor \frac{t+1}{4} \rfloor + 1$.*

Proof. Observe that, in the tessellations, the diamond vertices are all at mutual distance at most t, and that two vertices having the same relative position within two different diamonds of the tessellation are at distance greater than t. Hence, repeating the same vertex coloring for each diamond, the same color is repeated in the tessellation only at vertices whose distance is at least $t + 1$. Since H contains at least a whole diamond, the optimality follows by Lemma 5. □

In the following, it is shown how a color can be assigned in constant time to any vertex u of the grid, provided that u

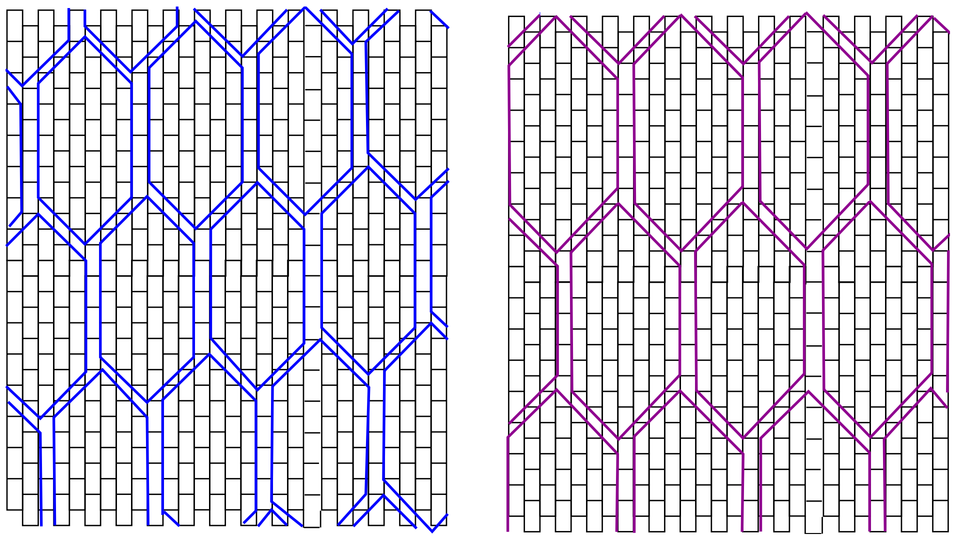

Fig. 8. The honeycomb tessellation: on the left, by the symmetric diamonds ($t = 13$); on the right, by the asymmetric diamonds ($t = 15$).

knows its coordinates (i, j) within the network. The coloring depends on the symmetry of the diamond.

4.1.1. *Coloring with symmetric diamonds in $O(1)$ time*

In this section, in order to achieve a simple, periodic, and arithmetic rule for coloring the grid vertices, a super-tessellation is imposed on the grid. Such a super-tessellation consists of large rectangles, each containing several diamonds.

Consider the symmetric case with $t = 8p + 1$ (the other symmetric case with $t = 8p+5$ can be dealt with similarly). In such a case, the diamond has as many right columns as its left columns, namely $2p$ (see the leftmost diamond in Fig. 7).

Observe the honeycomb tessellation by the symmetric diamonds, and restrict the attention to the rectangle $\mathcal{R}$, consisting of the leftmost $4p + 1$ columns and the uppermost $\omega(A_{H,8p+1}) = 24p^2 + 12p + 1$ rows of the grid, as depicted in Fig. 9 for $t = 9$ (namely $p = 1$). Clearly, the top left corner of $\mathcal{R}$ has coordinates (0,0). Sequentially scanning top-down the vertices in column 0 of $\mathcal{R}$, $4p + 1$ different diamonds are encountered. Moreover, for each traversed diamond, a different column is encountered and overall all the $4p + 1$ diamond columns, and hence all the diamond vertices, are met. By the above property, assigning a different color to each vertex in column 0 of $\mathcal{R}$ allows each diamond within the tessellation to be colored the same using the minimum number of colors.

To achieve such a goal, some technicalities are introduced below. First, let the diamond columns be numbered from left to right, starting from 0 and ending at $4p$. The diamond columns met along column 0 of $\mathcal{R}$ follow the order shown in Table 2. Formally, denoted by $x(j)$ the order in which the diamond column j is encountered, it holds that

$$x(j) = j(4p - 1) \bmod (4p + 1).$$

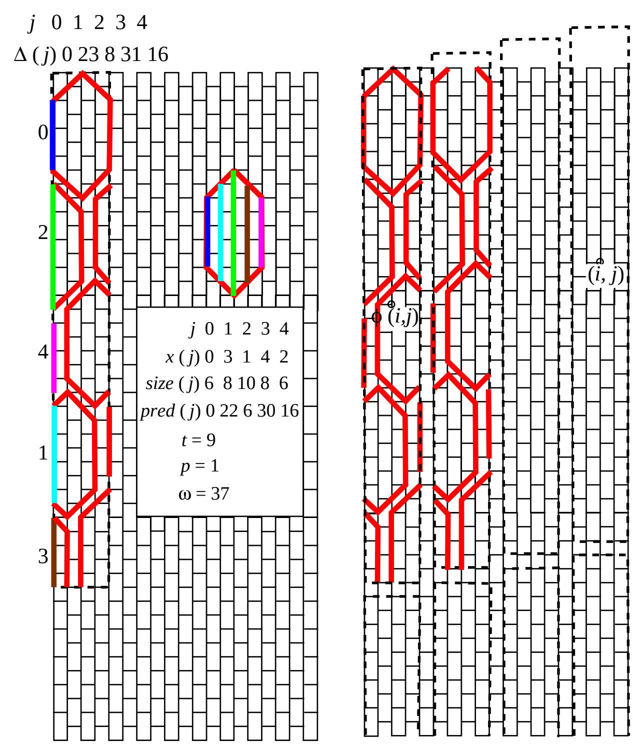

Fig. 9. The diamond columns encountered by scanning column 0 of $\mathcal{R}$ (left). The coloring of H by copies of $\mathcal{R}$ (right).

Conversely, given the order x in which a column is encountered, the column index $col(x)$ results to be

$$col(x) = 2px \bmod (4p + 1).$$

Moreover, the size, i.e. the number of vertices, of the diamond column j is given by

$$size(j) = \begin{cases} 4p + 2 + 2j & \text{if } 0 \leqslant j \leqslant 2p, \\ 12p + 2 - 2j & \text{if } 2p \leqslant j \leqslant 4p. \end{cases}$$

Table 2
The order in which diamond columns are encountered

Order	0	1	2	3	4	...	$4p-1$	$4p$
Column	0	$2p$	$4p$	$2p-1$	$4p-1$	...	1	$2p+1$

Finally, the number of vertices of a diamond that have been encountered before the topmost vertex of column j is

$$pred(j) = \sum_{k=0}^{x(j)-1} size(col(k)).$$

As an example, $x(j)$, $size(j)$ and $pred(j)$ are also shown in Fig. 9 (left) for $t=9$ (i.e. $p=1$).

Now, in order to assign different colors to all the vertices in column 0 of $\mathcal{R}$, let the color of vertex $(i,0)$ be simply $g(i,0)=i$. The coloring of the entire rectangle $\mathcal{R}$ is obtained assigning to the remaining columns a suitable cyclic shift of the coloring of column 0. Such a cyclic shift is chosen so that all the diamond columns with the same number are colored the same in all diamonds. To do this, let the shift for column j be denoted by $\Delta(j)$. Given the above coloring for column 0 of $\mathcal{R}$, it is easy to see that $\Delta(j)$, where $0 \leqslant j \leqslant 4p$, must be

$$\Delta(j) = \begin{cases} (pred(j)+2p)-(2p-j) & \text{if } 0 \leqslant j \leqslant 2p, \\ (pred(j)+2p)-(j-2p) & \text{if } 2p \leqslant j \leqslant 4p. \end{cases}$$

In conclusion, given any vertex $(i,j) \in \mathcal{R}$, its color is defined as

$$g(i,j) = (\Delta(j)+i) \bmod (24p^2+12p+1).$$

Note that, by construction, if $j=0$ then $g(i,0)=i$, while if $i=0$ then $g(0,j)=\Delta(j)$. The coloring of the entire grid H is obtained by defining a t-homomorphism $\phi : V(H) \mapsto V(\mathcal{R})$, which can be viewed as a covering of H with colored copies of $\mathcal{R}$, where the rectangles are shifted up by one row, as shown in Fig. 9 (right). Hence, for any vertex $(i,j) \in H$, its color $f(i,j)$ is given by $g(\phi(i,j))$, where

$$\phi(i,j) = \left(\left(i + \left\lfloor \frac{j}{4p+1} \right\rfloor \right) \bmod (24p^2+12p+1),\ j \bmod (4p+1) \right).$$

Observe that $\left\lfloor \frac{j}{4p+1} \right\rfloor$ counts how many rows the rectangle to which (i,j) belongs is shifted up with respect to the leftmost rectangle containing row i. Clearly, if $(i,j) \in \mathcal{R}$ then $\phi(i,j)=(i,j)$ and thus $f(i,j)=g(i,j)$. Observe also that $\mathcal{R}$ contains $O(p^3)$ vertices, and that the computation of each $\Delta(j)$ requires $O(p)$ time. Since t (and hence p) is a constant, the coloring of vertex (i,j) takes $O(1)$ time.

4.1.2. Coloring with asymmetric diamonds in $O(1)$ time

In this section, in order to achieve a simple, periodic, and arithmetic rule for coloring the grid vertices, a super-tessellation is again imposed on the grid. Such a super-tessellation, however, is much simpler than that introduced for the symmetric case.

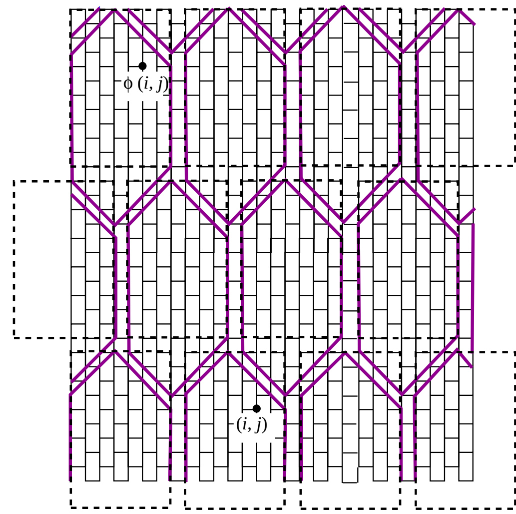

Fig. 10. The coloring of H by copies of $\mathcal{R}$ in the asymmetric case.

Consider now the asymmetric case with $t=8p+q$, where $q=3,7$. In such a case, the diamond has one more right column than its left columns (see Fig. 7). Due to the fact that the diamonds are horizontally aligned in the tessellation (see Fig. 8), the coloring is much simpler than in the symmetric case. Let *left* and *right* denote the number of left and right columns, respectively. As one can check in the proof of Lemma 5

$$left = \begin{cases} 2p & \text{if } q=3, \\ 2p+1 & \text{if } q=7 \end{cases}$$

and

$$right = \begin{cases} 2p+1 & \text{if } q=3, \\ 2p+2 & \text{if } q=7. \end{cases}$$

Observe the honeycomb tessellation by the asymmetric diamonds, and restrict the attention to the rectangle $\mathcal{R}$, consisting of the leftmost $left+right+1$ columns and the uppermost $t-left-1$ rows of the grid. As before, the top left corner of $\mathcal{R}$ is vertex (0,0).

The number of vertices in $\mathcal{R}$ is exactly $\omega(A_{H,t})$, where

$$\omega(A_{H,t}) = \begin{cases} 24p^2+24p+5 & \text{if } q=3, \\ 24p^2+48p+23 & \text{if } q=7. \end{cases}$$

Due to such a property, any coloring of the grid H obtained by covering H by colored copies of $\mathcal{R}$ (see Fig. 10) leads to a feasible and optimal coloring. Hence, one can use any coloring of $\mathcal{R}$ with as many colors as the number of vertices. For the sake of simplicity, let $\mathcal{R}$ be colored in row-major order. In details, given any vertex $(i,j) \in \mathcal{R}$, let its color be

$$g(i,j) = (i(left+right+1)+j) \bmod (\omega(A_{H,t})).$$

The coloring of the entire grid H is obtained by defining a t-homomorphism $\phi : V(H) \mapsto V(\mathcal{R})$. Precisely, for any vertex $(i, j) \in H$, its color $f(i, j)$ is given by $g(\phi(i, j))$ where

$$\phi(i, j) = \left(i \bmod (t - left), \left(j - right \left\lfloor \frac{i}{t - left} \right\rfloor \right) \bmod (left + right + 1) \right).$$

4.2. $L(1^t)$-coloring with t even

In this subsection, the coloring of the honeycomb grid is considered when t is even, that is when $t \equiv 0, 2, 4, 6 \bmod 8$. Again a lower bound on the size of a clique is presented which, however, is too weak. Therefore, a stronger lower bound based on the size of a t-independent set is stated in Section 4.2.1. Even in this case, two tessellations could be derived to optimally color the entire grid, based on properly "enlarged" diamonds. However, a much simpler way to color each vertex in $O(1)$ time is devised in Section 4.2.2. Such a method is based on a simple, periodic, and arithmetic rule which applies to all four even values $t \equiv 0, 2, 4, 6 \bmod 8$.

The following simple lower bound on the number of colors holds.

Lemma 6. *Let $t = 8p + q$, with $p \geqslant 0$ and $q = 0, 2, 4, 6$. There is an $L(1^t)$-coloring of a honeycomb grid H of size $r \times c$, with $r \geqslant t + 1$ and $c \geqslant \lfloor \frac{t}{4} \rfloor + \lceil \frac{t}{4} \rceil + 1$, only if*

$$\chi_t(H) \geqslant \begin{cases} 24p^2 + 6p + 1 & if\ q = 0, \\ 24p^2 + 18p + 4 & if\ q = 2, \\ 24p^2 + 30p + 10 & if\ q = 4, \\ 24p^2 + 42p + 19 & if\ q = 6. \end{cases}$$

Proof. As in Lemma 5, the maximum clique of $A_{H,t}$ is again a diamond, which can be symmetric or asymmetric. However, there are some *holes* (i.e., vertices not included in the clique) on a single border column of the diamond. The holes are located according to the *center* of the diamond. The center is the middle vertex (i, j) of the central column, which can be termed either *left center* or *right center* depending on whether it is horizontally connected either to vertex $(i, j-1)$ or $(i, j+1)$, respectively. In the symmetric case, the holes are located in the furthest column on the opposite side with respect to the horizontal connection of the center. Namely, if the clique has a left (resp., right) center then the holes are on the rightmost (resp., leftmost) column. Instead, in the asymmetric case, the holes are located on the same side as the center connection. Namely, if the clique has a left (resp., right) center then the holes are on the leftmost (resp., rightmost) column.

To compute the clique size $\omega(A_{H,t})$, consider first the subcase $t = 8p + q$, with $q = 0, 4$. In this case, since t is a multiple of 4, the diamond consists of a central column of $t+1$ vertices, $\frac{t}{4}$ right columns, and $\frac{t}{4}$ left columns. Moreover, there are $h = \frac{t}{4}$ holes. In particular, depending on the value of q,

$$\frac{t}{4} = \begin{cases} 2p & \text{if } q = 0, \\ 2p + 1 & \text{if } q = 4. \end{cases}$$

The size of the maximum clique is given by

$$\omega(A_{H,t}) = (t + 1) + 2 \sum_{i=1}^{t/4} (t + 1 - 2i) - \frac{t}{4}.$$

Solving the above formula with $t = 8p$ yields $\omega(A_{H,t}) = 24p^2 + 6p + 1$. Similarly, when $t = 8p + 4$, $\omega(A_{H,t}) = 24p^2 + 30p + 10$ holds.

Consider now the subcase $t = 8p + q$, with $q = 2, 6$. In this case, the diamond is asymmetric and includes, in addition to the central column, $\lceil \frac{t}{4} \rceil$ columns on the same side as the horizontal connection of the clique center, and $\lfloor \frac{t}{4} \rfloor$ columns on the opposite side. Hence, in the furthest column, there are $h = \lfloor \frac{t}{4} \rfloor$ holes. Depending on the value of q,

$$\left\lceil \frac{t}{4} \right\rceil = \begin{cases} 2p + 1 & \text{if } q = 2, \\ 2p + 2 & \text{if } q = 6 \end{cases}$$

and

$$\left\lfloor \frac{t}{4} \right\rfloor = \begin{cases} 2p & \text{if } q = 2, \\ 2p + 1 & \text{if } q = 6. \end{cases}$$

Hence, the size of the maximum clique is given by

$$\omega(A_{H,t}) = (t + 1) + \sum_{i=1}^{\lceil \frac{t}{4} \rceil} (t + 1 - 2i) - \frac{t}{4} + \sum_{i=1}^{\lfloor \frac{t}{4} \rfloor} (t + 1 - 2i) - \left\lfloor \frac{t}{4} \right\rfloor.$$

Solving the above formula with $t = 8p + 2$, one has $\omega(A_{H,t}) = 24p^2 + 18p + 4$, while when $t = 8p + 4$, $\omega(A_{H,t}) = 24p^2 + 42p + 19$.

As an example, Fig. 11 shows the maximum cliques when $q = 0, 2, 4$ and 6 and $t = 16, 18, 20$ and 22, respectively. In such a figure, the clique centers are depicted by black dots. □

4.2.1. A stronger lower bound based on t-independent sets

In contrast to the case t odd, when t is even the lower bound on the number of colors given by $\omega(A_{H,t})$ is no more reachable by an $L(1^t)$-coloring. Indeed it is possible to derive a stronger lower bound based on the notion of t-independent set.

Thus, consider now how a maximum t-independent set S_t^* can be built. First observe that, in the case of t even, due to the asymmetry of the horizontal connections, it is not possible to choose three vertices pairwise at distance $t + 1$.

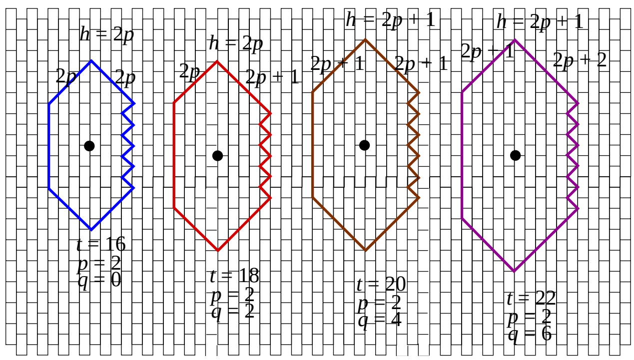

Fig. 11. The maximum cliques (diamonds) for $t = 16, 18, 20$ and 22. The number of holes is denoted by h.

Lemma 7. *When t is even, the minimum distances among three closest vertices belonging to S_t^* are $t+1, t+1$ and $t+2$.*

Proof. Let a vertex $u = (i, j)$ of H be a *left vertex*, if it is horizontally connected to vertex $s = (i, j-1)$, or a *right vertex* if it is connected to $d = (i, j+1)$.

By contradiction, assume there are 3 vertices u, v and w such that $d(u, v) = d(v, w) = d(w, u) = t + 1$. W.l.o.g., let u be a left vertex. Since $t + 1$ is odd, then both v and w must be right vertices. This implies that $d(v, w)$ must be even and greater than t. But this is a contradiction, and $d(v, w) \geqslant t + 2$. □

Given a vertex v, all the vertices at distance at most t from v cannot belong to the same t-independent set. Let $B_t(v)$ be the set of vertices at distance exactly $t+1$ from v. It is easy to show that $|B_t(v)| = 3t + 3$ for any t even, as depicted in Fig. 12 for the symmetric (left) and asymmetric (right) cases.

To build a maximum t-independent set that contains v, let select as many vertices as possible among those in $B_t(v)$. By Lemma 7, since those vertices are all at distance $t + 1$ from v, they must be at distance at least $t + 2$ among them.

Lemma 8. *When t is even, there is no way to select 6 vertices $u_0, u_1, \ldots, u_5$ of $B_t(v)$ such that $d(u_i, u_{(i+1) \bmod 6}) = t + 2$.*

Proof. Since any two consecutive vertices of $B_t(v)$ are at distance 2, no more than one out of $\frac{t+2}{2}$ consecutive vertices of $B_t(v)$ can be selected. Therefore at most $\left\lfloor \frac{3t+3}{\frac{t+2}{2}} \right\rfloor = 5$ vertices can be selected. □

The following result also holds.

Lemma 9. *When t is even, there is no way to select 6 vertices $u_0, u_1, \ldots, u_5$ such that*

- *u_i belongs to $B_t(v)$ for $i = 1, \ldots, 5$;*
- *$d(u_i, u_{i+1}) = t + 2$, for $i = 1, \ldots, 4$;*
- *$d(u_0, u_1) = d(u_0, u_5) = t + 1$.*

Proof. After selecting $u_1, \ldots, u_5$ on $B_t(v)$ such that $d(u_i, u_{i+1}) = t + 2$ for $i = 1, \ldots, 4$, there are $t - 2$ vertices of $B_t(v)$ left out between u_5 and u_1 and t vertices at distance $t + 2$ from v between u_5 and u_1 (see the squares in Fig. 13). Since by Lemma 8, $d(u_0, v) \geqslant t + 2$, there is no way to choose any vertex u_0 at distance $t + 1$ from both u_0 and u_5 and at distance $t + 2$ from v. □

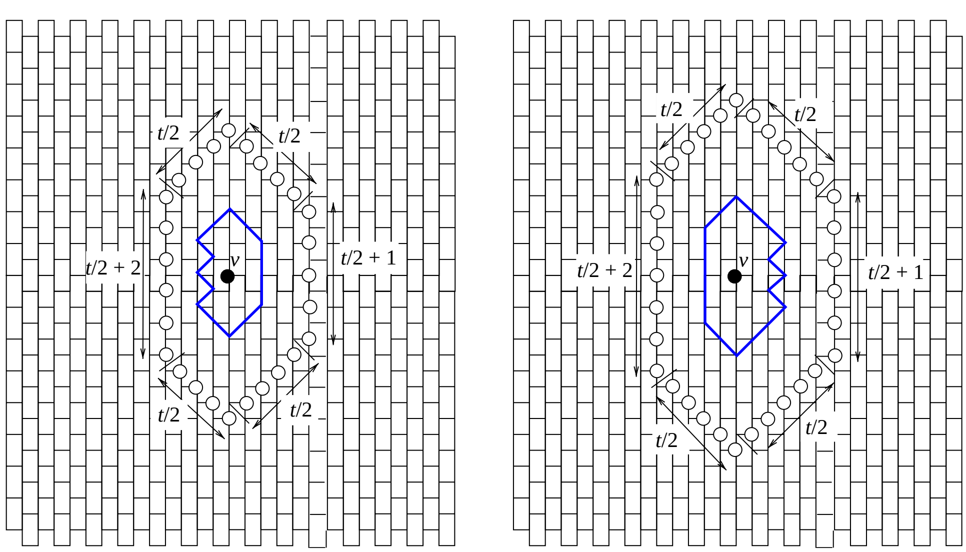

Fig. 12. The set $B_t(v)$ of $3t + 3$ vertices at distance exactly $t + 1$ from vertex v: symmetric case (left), asymmetric case (right).

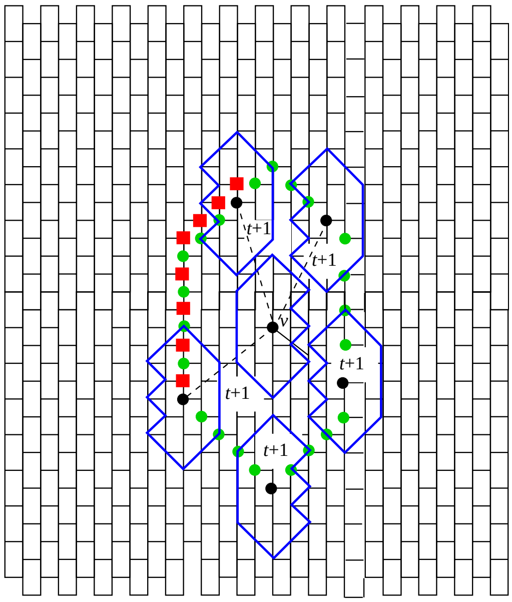

Fig. 13. A vertex v and its set $B_t(v)$ of vertices (denoted by circles) at distance $t+1$ from v. Chosen $u_1, \ldots, u_5$ in $B_t(v)$ (depicted by black circles), the possible candidates for u_0 at distance $t+2$ from v are denoted by squares.

In conclusion, by the previous lemmas, to build a maximum t-independent set including a given vertex v, one should choose the six vertices closest to v such that at most four of them are at distance $t+1$ from v, and at least two of them are at distance $t+2$. This is possible for any t even, as shown for instance in Fig. 14 when $t=8$.

Moreover, in a maximum t-independent set such a property should hold for any of its vertices, and in particular for the 6 vertices closest to v.

Lemma 10. *Let $t = 8p + q$, with $p \geqslant 0$ and $q = 0, 2, 4, 6$. The minimum number of maximum t-independent sets that cover a sufficiently large honeycomb grid H is*

$$\mu_t(H) \geqslant \begin{cases} 24p^2 + 8p + 1 & if\ q = 0, \\ 24p^2 + 20p + 4 & if\ q = 2, \\ 24p^2 + 32p + 11 & if\ q = 4, \\ 24p^2 + 44p + 20 & if\ q = 6. \end{cases}$$

Proof. Choose a vertex v of S_t^* and its 6 closest vertices such that 4 of them are at distance $t+1$ and the remaining 2 are at distance $t+2$. By Lemma 6, each of these vertices can be perceived as a center of a diamond. The vertices of each diamond must belong all to different independent sets because they are pairwise at distance at most t. Building such a diamond around each vertex of S_t^*, one obtains a tessellation of H with some *uncovered* vertices between any two diamonds whose centers are at distance $t+2$. A possible placement of the vertices of the maximum t-independent set is depicted in Fig. 15 for $t=8$ (on the left) and $t=10$ (on the right). Note that there is no way to decrease the number of uncovered vertices because, by Lemmas 8 and 9, v and its 6 closest vertices are as dense as possible. Clearly, these uncovered vertices cannot belong to S_t^*.

Since every center has two closest vertices at distance $t+2$, by observing Fig. 15, one notes that there are two groups of uncovered vertices adjacent to any diamond: one group is above the diamond left columns and the other group is below the diamond right columns. Now, a mapping can be defined between each diamond center and the uncovered vertices by assigning to a diamond with a left center the uncovered vertices above its left columns, and assigning to a diamond with a right center the uncovered vertices above its right columns. Note that if the two diamond centers are at distance $t+2$, they are both either left or right centers. Hence, the mapping assigns the uncovered vertices between the two diamonds to one and only one of them.

Finally, the number of uncovered vertices associated to each diamond is the minimum between the number of left and right columns of the diamond, which in turn is exactly equal the number h of holes of the diamond:

$$h = \begin{cases} 2p & \text{if } q = 0, 2, \\ 2p+1 & \text{if } q = 4, 6. \end{cases}$$

Hence, only one vertex out of the diamond vertices and its holes, that is one out of $\omega(A_{H,t}) + h$ vertices, can belong to S_t^*. Therefore, the minimum number $\mu_t(H)$ of maximum t-independent sets needed to cover the grid H, which must be large enough to contain 7 diamonds and their uncovered vertices (see Fig. 14), is at least

$$\omega(A_{H,t}) + h = \begin{cases} 24p^2 + 8p + 1 & \text{if } q = 0, \\ 24p^2 + 20p + 4 & \text{if } q = 2, \\ 24p^2 + 32p + 11 & \text{if } q = 4, \\ 24p^2 + 44p + 20 & \text{if } q = 6. \end{cases} \qquad \square$$

4.2.2. Optimal coloring in $O(1)$ time

By Lemma 10, to derive optimal colorings when t is even, one could consider a diamond enlarged in such a way that it includes also all its h holes. Then one could tessellate the grid by means of the enlarged diamonds, using exactly the same techniques already seen in the case that t is odd.

However, when $t = 8p+q$ is even, there is a simple, periodic, and arithmetic rule for optimal $L(1^t)$-coloring honeycomb grids. Such a rule assigns in $O(1)$ time to each vertex $u = (i, j)$ the color

$$f(i, j) = (\alpha i + \beta j) \bmod \chi_t$$

where the proper values of the parameters α and β, for $q = 0, 2, 4,$ and 6, are shown in Table 3.

Theorem 4. *When t is even, the above arithmetic rule leads to a feasible $L(1^t)$-coloring for any honeycomb grid H. Such an $L(1^t)$-coloring is optimal when H is large enough to contain 7 diamonds and their uncovered vertices.*

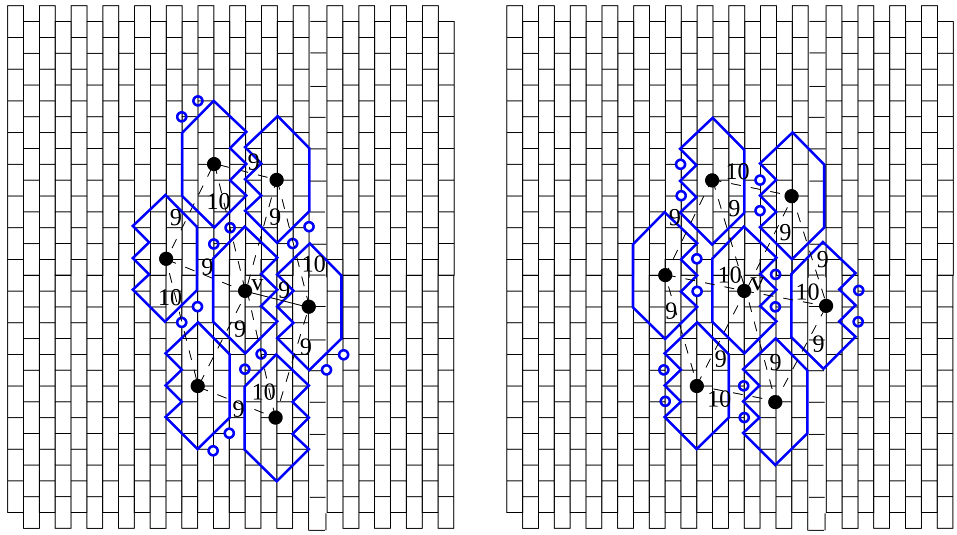

Fig. 14. Two ways of selecting the 6 closest vertices to a vertex v in a maximum t-independent set.

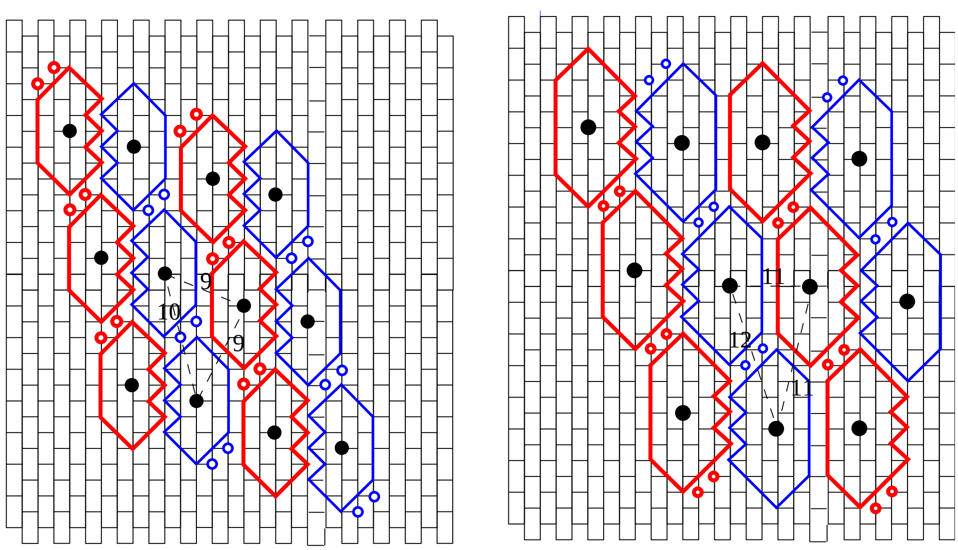

Fig. 15. The maximum 8-independent set (left) and 10-independent set (right) consisting of the diamond centers (depicted by black dots). The uncovered vertices are shown by white circles.

Proof. In the following, it is shown that for any colored vertex $u = (i, j)$, its closest vertices $v = (i', j')$ colored the same appear at distance $d(u, v) \geqslant t + 1$. For this purpose, the following equation:

$$\alpha|i' - i| + \beta|j' - j| \equiv_{\chi_t} 0 \qquad (1)$$

is solved for $t = 8p + q$ and $q = 0, 2, 4$ and 6, where α, β and χ_t are given in Table 3. Recall that $d(u, v)$ in the honeycomb grid is at least equal to the Manhattan distance $|i' - i| + |j' - j|$.

Case 1: When $q = 0$, the above Eq. (1) becomes $|i' - i| + (4p + 1)|j' - j| \equiv_{24p^2+8p+1} 0$. Solving such an equation, one has the following six vertices: $v_1 = (i - 2p, j + 6p + 1)$, $v_2 = (i + 2p, j - 6p - 1)$, $v_3 = (i + 2p + 1, j + 6p)$, $v_4 = (i - 2p - 1, j - 6p)$, $v_5 = (i + 4p + 1, j - 1)$, and $v_6 = (i - 4p - 1, j + 1)$. By the Manhattan distance, it is easy to see that $d(u, v_i) \geqslant 8p + 1$, with $1 \leqslant i \leqslant 4$. Moreover, $d(u, v_5) = d(u, v_6) = \lfloor \frac{4p+1}{2} \rfloor 4 + 1 = 8p + 1$.

Case 2: When $q = 2$, Eq. (1) is $(6p + 2)|i' - i| + (2p + 1)|j' - j| \equiv_{24p^2+20p+4} 0$. The solutions of the equation are: $v_1 = (i - 2p - 1, j - 6p - 2)$, $v_2 = (i + 2p + 1, j + 6p + 2)$, $v_3 = (i - 2p - 1, j + 6p + 2)$, $v_4 = (i + 2p + 1, j - 6p - 2)$, $v_5 = (i + 4p + 2, j)$, and $v_6 = (i - 4p - 2, j)$. By the Manhattan distance, one had $d(u, v_i) = 8p + 3$, when $1 \leqslant i \leqslant 4$. Finally, $d(u, v_5) = d(u, v_6) = 4\frac{4p+1}{2} = 8p + 4$.

Case 3: When $q = 4$, one must solve $|i' - i| + (4p + 3)|j' - j| \equiv_{24p^2+32p+11} 0$, whose solutions are: $v_1 = (i - 2p - 2, j - 6p - 3)$, $v_2 = (i + 2p + 2, j + 6p + 3)$, $v_3 =$

Table 3
The values of the parameters in the rule $f(i, j) = (\alpha i + \beta j) \bmod \chi_{8p+q}$

q	α	β	χ_{8p+q}
0	1	$4p+1$	$24p^2+8p+1$
2	$6p+2$	$2p+1$	$24p^2+20p+4$
4	1	$4p+3$	$24p^2+32p+11$
6	$6p+5$	$2p+2$	$24p^2+44p+20$

$(i-2p-1, j+6p+4)$, $v_4 = (i+2p+1, j-6p-4)$, $v_5 = (i+4p+3, j-1)$, and $v_6 = (i-4p-3, j+1)$. When $1 \leqslant i \leqslant 4$, $d(u, v_i) \geqslant 8p+5$, while $d(u, v_5) = d(u, v_6) = 8p+5$.

Case 4: When $q = 6$, one obtains $(6p+5)|i'-i|+(2p+2)|j'-j| \equiv_{24p^2+44p+20} 0$. The closest vertices colored the same are: $v_1 = (i+2p+2, j-6p-5)$, $v_2 = (i-2p-2, j+6p+5)$, $v_3 = (i+2p+2, j+6p+5)$, $v_4 = (i-2p-2, j-6p-5)$, $v_5 = (i+4p+4, j)$, and $v_6 = (i-4p-4, j)$. When $1 \leqslant i \leqslant 4$, $d(u, v_i) \geqslant 8p+7$, while $d(u, v_5) = d(u, v_6) = 8p+8$.

Note that in all the above cases, there are always four vertices exactly at distance $8p+q+1$ and two vertices at distance $8p+q+2$, as a consequence of Lemmas 8 and 9.

The optimality follows from Lemma 10, observed that H must be sufficiently large to contain at least one of the configurations depicted in Fig. 14. □

5. Conclusion

This paper has considered a graph theoretical approach for the $L(2,1)$-, $L(2,1^2)$-, and $L(1^t)$-coloring problems on honeycomb grids. Such coloring problems model the minimum-span fixed channel assignment (FCA) problem on a flat region without geographical barriers, where the wireless network base stations, placed according to a plane tessellation based on regular hexagons, receive a single channel per station. Precisely, after recalling some preliminary graph theoretical results, simple, periodic, and arithmetic rules were presented to optimally solve the $L(2,1)$-, $L(2,1^2)$-, and $L(1^t)$-coloring problems. While the solutions for the first two cases have been readily derived, a much more complicated strategy has been followed to obtain the optimal solution for $L(1^t)$-coloring.

The results derived in this paper are summarized in the first row of Table 4. Such table also indicates the minimum number of channels required for the square and cellular grids. Hence, overall the table summarizes the results, known up to now in the literature, for the three grids which correspond to all the plane tessellations based on regular polygons. In all the cases, there are efficient algorithms to assign channels to vertices. The channel assigned to any vertex can be computed locally provided that the relative position of the vertex in the network is known. Such a computation can be performed in constant time for all the networks.

Table 4
Minimum number of channels used for a sufficiently large network G (for honeycomb grids, $t = 8p+q$)

Network G	$L(1^t)$	$L(2,1)$	$L(2,1^2)$
Honeycomb grid	$\begin{cases} 24p^2+8p+1 & \text{if } q=0 \\ 24p^2+12p+2 & \text{if } q=1 \\ 24p^2+20p+4 & \text{if } q=2 \\ 24p^2+24p+6 & \text{if } q=3 \\ 24p^2+32p+11 & \text{if } q=4 \\ 24p^2+36p+14 & \text{if } q=5 \\ 24p^2+44p+20 & \text{if } q=6 \\ 24p^2+48p+24 & \text{if } q=7 \end{cases}$	6	7
Square grid	$\lceil \frac{1}{2}(t+1)^2 \rceil$	7	9
Cellular grid	$\lceil \frac{3}{4}(t+1)^2 \rceil$	9	12
References	[4,12]	[5,8,12,24]	[5,24]

By observing Table 4, one notes that, for any $t > 1$, the proposed colorings for honeycomb grids use less colors than those needed by the cellular and square grids. For instance, considering $t = 8p+q$ and deriving the minimum number of channels for square grids, one obtains

$$\left\lceil \frac{(t+1)^2}{2} \right\rceil = \left\lceil 32p^2 + 8p(q+1) + \frac{(q+1)^2}{2} \right\rceil,$$

which is greater than the number of colors used for honeycomb grids whenever $t > 1$.

The solutions in Table 4 assume that a single channel has to be assigned to each station. However, by standard techniques, the proposed solutions can be readily generalized to derive sub-optimal solutions for uniform multi-channel assignment and hybrid channel assignment. Indeed, when the same number m of channels has to be assigned to each vertex, the optimal solution here proposed can be extended as follows. Assume that s colors are used in total and that a vertex gets the color i, then such a vertex receives also colors $i+s, i+2s, \ldots, i+(m-1)s$. Moreover, such a uniform multi-channel solution can be used to determine the channels in the fixed set used by a HCA strategy. Finally, additional work is needed to extend the present solution to the FCA with borrowing as well as to the DCA strategies. For instance, in DCA, the channels are often partitioned into groups, while the base stations are partitioned into clusters. Base stations can try in a distributed way to get a free channel group that is not held by one of its neighbors [7]. Usually groups have no structure other than to be a set of disjoint channels. Our approach can provide groups with guaranteed separations among the channels in the group in order to help base stations to dynamically select the channels to be used within the clusters.

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